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Isotropic Elastic Solids of nearly Spherical Form.

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Part I.—Equilibrium.

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§1. The investigation of a solution of the elastic solid equations for the equilibrium or motion of homogeneous isotropic material enclosed by the simplest of all surfaces, the spherical, presents no small difficulty. For even a slight departure from the spherical form the increase of difficulty is so considerable that, so far as I know, the only problem of the class successfully treated hitherto is that of a nearly spherical solid exposed to gravitational forces but free of all surface forces.[*] In the present case, surface forces appear as well as bodily forces, so that the problem is much more general than that previously treated. The method employed is also novel, and seems likely to prove serviceable in various other branches of applied mathematics.

Notation.

§2. Polar coordinates r, θ, ϕ are employed, θ being measured from the "polar axis." The displacements u, v, w are measured as in Todhunter and

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[*] Cambridge Philosophical Society's Transactions, Vol. XIV, p. 278 *et seq.*

Pearson's "History," and the convenient notation of the same work is also adopted for the stresses. The expressions for the strains and the body-stress equations—or equations satisfied by the stresses throughout the interior—may be deduced from the footnote, p. 79, Vol. II, Part I, of Todhunter and Pearson's "History" by writing θ, ϕ for ϕ and ψ respectively. The dilatation is denoted by Δ, so that

$$\Delta = \frac{du}{dr} + \frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi}.$$

The density is denoted by ρ and the elastic constants by m and n, as used in Thomson and Tait's "Natural Philosophy."

§3. The present paper employs the general solution in polar coordinates, involving surface spherical harmonics, given in Vol. XIV of the Cambridge "Transactions," p. 250, but with a slightly different notation. We need write down only the terms depending on surface harmonics of a single degree, i, as types. Thus we have

$$\Delta = \frac{-\rho}{m+n} r^i V_i \mathbf{V}_i + r^i Y_i \mathbf{Y}_i, \tag{1}$$

$$u = -r^{i+1}\left\{\frac{(i+2)\rho V_i \mathbf{V}_i}{2(2i+3)(m+n)} + \frac{im-2n}{2(2i+3)n} Y_i \mathbf{Y}_i\right\} + r^{i-1} Z_i \mathbf{Z}_i, \tag{2}$$

$$v = \frac{d}{d\theta}\left[-\frac{r^{i+1}}{2(2i+3)}\left\{\frac{\rho}{m+n} V_i \mathbf{V}_i + \frac{(i+3)m+2n}{(i+1)n} Y_i \mathbf{Y}_i\right\} + \frac{r^{i-1}}{i} Z_i \mathbf{Z}_i\right] + \frac{1}{\sin\theta}\frac{d}{d\phi}(r^i W_i \mathbf{W}_i), \tag{3}$$

$$w = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{r^{i+1}}{2(2i+3)}\left\{\frac{\rho}{m+n} V_i \mathbf{V}_i + \frac{(i+3)m+2n}{(i+1)n} Y_i \mathbf{Y}_i\right\} + \frac{r^{i-1}}{i} Z_i \mathbf{Z}_i\right] - \frac{d}{d\theta}(r^i W_i \mathbf{W}_i). \tag{4}$$

Here V_i, Y_i, etc., denote constants, while $\mathbf{V}_i$, $\mathbf{Y}_i$, etc., represent each one or a series of surface harmonics. The bodily forces come from the potential $r^i V_i \mathbf{V}_i$, so that the form of the harmonic, or harmonics, $\mathbf{V}_i$ is given. The forms of $\mathbf{Y}_i$, $\mathbf{Z}_i$, $\mathbf{W}_i$ depend partly on $\mathbf{V}_i$ and partly on the surface harmonics which occur in the expression for the surface forces.

When the solution is applied to a perfect sphere, $\mathbf{Y}_i$ and $\mathbf{Z}_i$ may be treated quite separately from $\mathbf{W}_i$. The latter depends only on surface forces of such a type that

$$\text{Component parallel to } \theta = \frac{1}{\sin\theta}\,\frac{d}{d\phi}f_1(\theta,\phi),$$
$$\text{" \quad " \quad } \phi = -\frac{d}{d\theta}f_1(\theta,\phi),$$

where $f_1(\theta,\phi)$ denotes a function of θ and ϕ, which may be spoken of as a "stream function."

On the other hand, in a perfect sphere $\mathbf{Y}_i$ and $\mathbf{Z}_i$ depend partly on $\mathbf{V}_i$ and partly on surface forces of such a type that

$$\text{Component parallel to } \theta = \frac{d}{d\theta}f_2(\theta,\phi),$$
$$\text{" \quad " \quad } \phi = \frac{1}{\sin\theta}\,\frac{d}{d\phi}f_2(\theta,\phi),$$

where $f_2(\theta,\phi)$ is a function of θ and ϕ, which may be spoken of as a "potential function."

When the solution is applied to a nearly spherical mass, this separation of $\mathbf{Y}_i$ and $\mathbf{Z}_i$ from $\mathbf{W}_i$ is not in general possible.

§4. The arbitrary constants Y_i, Z_i, W_i in the general solution have to be determined from the three surface equations, or equations satisfied by the stresses at the surface. Denoting the direction cosines of the outwardly directed normal relative to the fundamental directions r, θ, ϕ by λ, μ, ν, and employing R, Θ, Φ to signify the components of the applied surface forces in these directions, we have for surface equations

$$\left.\begin{aligned}\lambda\,\widehat{rr} + \mu\,\widehat{r\theta} + \nu\,\widehat{r\phi} &= R,\\ \lambda\,\widehat{r\theta} + \mu\,\widehat{\theta\theta} + \nu\,\widehat{\theta\phi} &= \Theta,\\ \lambda\,\widehat{r\phi} + \mu\,\widehat{\theta\phi} + \nu\,\widehat{\phi\phi} &= \Phi.\end{aligned}\right\} \qquad (5)$$

Here the values of the stresses $\widehat{rr}$, etc., answer of course to that point on the surface to which $\lambda, \ldots . R, \ldots .$ belong.

§5. In the method of solution about to be used we require the values of the stresses answering to the displacements (2), (3), (4) over the spherical surface $r \parallel a$. To secure brevity in the expressions for these stresses, let us put

$$\frac{-a^i}{2i+3}\left\{(i^2-i-3)m+n\right\} = A_i, \tag{6}$$

$$2(i-1)a^{i-2}n = B_i, \tag{7}$$

$$\frac{-a^i\{i(i+2)m-n\}}{(i+1)(2i+3)} = C_i, \tag{8}$$

$$2\frac{(i-1)}{i}a^{i-2}n = D_i, \tag{9}$$

$$\frac{a^i}{2i+3}\{(i+3)m-(2i+1)n\} = E_i, \tag{10}$$

$$2a^{i-2}n = F_i, \tag{11}$$

$$\frac{-a^i\{(i+3)m+2n\}}{(i+1)(2i+3)} = G_i, \tag{12}$$

$$\frac{2a^{i-2}}{i}n = H_i, \tag{13}$$

$$\frac{-\rho}{m+n}\,\frac{a^i}{2i+3}\left\{(2i+3)m+(i^2+i-1)n\right\} = L_i, \tag{14}$$

$$\frac{-\rho n}{m+n}\,a^i\,\frac{i+1}{2i+3} = M_i, \tag{15}$$

$$\frac{-\rho}{m+n}\,\frac{a^i}{2i+3}\{(2i+3)m-(i+1)n\} = N_i, \tag{16}$$

$$\frac{-\rho n}{m+n}\,\frac{a^i}{2i+3} = O_i. \tag{17}$$

Also denote $V_i\mathbf{V}_i$ by $[V_i]$, etc.

Then when $r=a$ we have

$$\widehat{rr} = L_i[V_i]+A_i[Y_i]+B_i[Z_i], \tag{18}$$

$$\widehat{\theta\theta} = N_i[V_i]+E_i[Y_i]+F_i[Z_i]+\frac{d^2}{d\theta^2}\{O_i[V_i]+G_i[Y_i]+H_i[Z_i]\} \\ +2na^{i-1}W_i\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\,\frac{d\mathbf{W}_i}{d\phi}\right\}, \tag{19}$$

$$\widehat{\phi\phi} = N_i[V_i]+E_i[Y_i]+F_i[Z_i] \\ +\left(\frac{1}{\sin^2\theta}\,\frac{d^2}{d\phi^2}+\cot\theta\,\frac{d}{d\theta}\right)\{O_i[V_i]+G_i[Y_i]+H_i[Z_i]\} \\ -2na^{i-1}W_i\frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\,\frac{d\mathbf{W}_i}{d\phi}\right\}, \tag{20}$$

$$\widehat{r\theta} = \frac{d}{d\theta}\{M_i[V_i] + C_i[Y_i] + D_i[Z_i]\} + n(i-1)a^{i-1}W_i\frac{1}{\sin\theta}\frac{d\mathbf{W}_i}{d\phi}, \tag{21}$$

$$\widehat{r\phi} = \frac{1}{\sin\theta}\frac{d}{d\phi}\{M_i[V_i] + C_i[Y_i] + D_i[Z_i]\} - n(i-1)a^{i-1}W_i\frac{d\mathbf{W}_i}{d\phi}, \tag{22}$$

$$\widehat{\theta\phi} = \frac{d^2}{d\theta d\phi}\{\operatorname{cosec}\theta(O_i[V_i] + G_i[Y_i] + H_i[Z_i])\} - na^{i-1}W_i\left\{2\frac{d^2\mathbf{W}_i}{d\theta^2} + i(i+1)\mathbf{W}_i\right\}. \tag{23}$$

When, as may frequently happen,

$$\mathbf{V}_i = \mathbf{Y}_i = \mathbf{Z}_i = T_i \text{ say,}$$

it is convenient to put

$$\left.\begin{aligned} L_iV_i + A_iY_i + B_iZ_i &= \overline{L}_i, \\ M_iV_i + C_iY_i + D_iZ_i &= \overline{M}_i, \\ N_iV_i + E_iY_i + F_iZ_i &= \overline{N}_i, \\ O_iV_i + G_iY_i + H_iZ_i &= \overline{O}_i. \end{aligned}\right\} \tag{24}$$

In such a case, when the equation to the surface also involves T_i, being of the form

$$r = a(1 + \varepsilon T_i),$$

where ε^2 is negligible, the following relations will be found useful:

$$\left.\begin{aligned} &\frac{dT_i}{d\theta}\frac{d^2T_i}{d\theta^2} + \frac{1}{\sin\theta}\frac{dT_i}{d\phi}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta T_i) = \tfrac{1}{2}\frac{d}{d\theta}\left\{\left(\frac{dT_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\right)^2\right\}, \\ &\frac{dT_i}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta T_i) + \frac{1}{\sin\theta}\frac{dT_i}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d^2T_i}{d\phi^2} + \cot\theta\frac{dT_i}{d\theta}\right) \\ &\qquad = \tfrac{1}{2}\frac{1}{\sin\theta}\frac{d}{d\phi}\left\{\left(\frac{dT_i}{d\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{dT_i}{d\phi}\right)^2\right\}. \end{aligned}\right\} \tag{25}$$

§6. The first problem I propose to treat is as follows: The isotropic elastic solid of density ρ, the equation to whose surface is

$$r = a(1 + \varepsilon P_2), \tag{26}$$

where P_2 is the second zonal harmonic and ε is a constant, positive or negative, whose square is negligible, is acted on by bodily forces derived from the potential

$$r^iV_iP_i,$$

and by the following surface forces:

$$\begin{array}{lcl} R_iP_i & \text{along} & r, \\ \Theta_i\dfrac{dP_i}{d\theta} & \text{"} & \theta, \\ -\Phi_i\dfrac{dP_i}{d\theta} & \text{"} & \phi, \end{array}$$

where P_i is the i^{th} zonal harmonic and V_i, R_i, Θ_i, Φ_i given finite constant quantities; to determine the elastic displacements, assuming equilibrium to exist.

The surface may be taken to represent any prolate or oblate spheroid of small eccentricity, and since i may have any positive integral value, the forces may be supposed to represent any continuous system symmetrical round the axis of the spheroid.

§7. Since terms in ε^2 are neglected, the direction cosines of the normal at a point on the surface relative to the fundamental directions r, θ, ϕ at the point are respectively 1, $-\varepsilon\dfrac{dP_2}{d\theta}$ and 0. Also the surface value of any function $f(r)$ occurring in the solution is given by

$$f(r)=f(a)+\varepsilon P_2 a\frac{df(a)}{da}.$$

In the first surface condition in the value of $\widehat{rr}$ we write $a+\varepsilon aP_2$ for r, but in $\widehat{r\theta}$ which has already ε in its coefficient we simply write a for r. Noticing such little points, and employing Σ to denote summation with respect to i, we may write the surface conditions, in the general case of a series of zonal harmonics, in the form

$$\Sigma\left(1+\varepsilon P_2 a\frac{d}{da}\right)\overline{L}_iP_i-\varepsilon\frac{dP_2}{d\theta}\Sigma\overline{M}_i\frac{dP_i}{d\theta}=\Sigma R_iP_i, \tag{27}$$

$$\Sigma\left(1+\varepsilon P_2 a\frac{d}{da}\right)\overline{M}_i\frac{dP_i}{d\theta}-\varepsilon\frac{dP_2}{d\theta}\Sigma\left(\overline{N}_iP_i+\overline{O}_i\frac{d^2P_i}{d\theta^2}\right)=\Sigma\Theta_i\frac{dP_i}{d\theta}, \tag{28}$$

$$\begin{aligned} -\Sigma(i-1)a^{i-1}W_i\frac{dP_i}{d\theta}+\varepsilon\frac{dP_2}{d\theta}\Sigma a^{i-1}W_i\left\{i(i+1)P_i+2\frac{d^2P_i}{d\theta^2}\right\}& \\ =-\frac{1}{n}\Sigma\Phi_i\frac{dP_i}{d\theta},& \end{aligned} \tag{29}$$

where $\overline{L}_i$, etc., are given by (24).

In such a differential coefficient as $\dfrac{d\overline{L}_i}{da}$ in (27) we are to treat Y_i and Z_i as absolute constants, and after differentiation—since ε occurs as a coefficient—we are to substitute the values they would have in a perfect sphere, i. e. the first approximations as obtained by putting $\epsilon = 0$ in the surface equations.

§8. The solution of the above equations, in which Y_i, Z_i, W_i represent the unknowns, presents no great difficulty when we have expressed:

In the first surface equation P_2P_i and $\frac{dP_2}{d\theta}\,\frac{dP_i}{d\theta}$ as a finite series of surface harmonics.

In the second surface equation $P_2\frac{dP_i}{d\theta}$, $P_i\frac{dP_2}{d\theta}$ and $\frac{dP_2}{d\theta}\,\frac{d^2P_i}{d\theta^2}$ as a finite series of terms of the form $\frac{dP_j}{d\theta}$.

In the third surface equation $P_i\frac{dP_2}{d\theta}$ and $\frac{dP_2}{d\theta}\,\frac{d^2P_i}{d\theta^2}$ as a finite series of terms of the form $\frac{dP_j}{d\theta}$.

These several objects are attained in the following formulae:

$$P_2P_i = \frac{3(i-1)i}{2(2i-1)(2i+1)}P_{i-2} + \frac{i(i+1)}{(2i-1)(2i+3)}P_i + \frac{3(i+1)(i+2)}{2(2i+1)(2i+3)}P_{i+2}, \tag{30}$$

$$\frac{dP_2}{d\theta}\,\frac{dP_i}{d\theta} = \frac{3i(i+1)}{(2i-1)(2i+1)(2i+3)}\left\{(i-1)(2i+3)P_{i-2} + (2i+1)P_i - (i+2)(2i-1)P_{i+2}\right\}, \tag{31}$$

$$P_2\frac{dP_i}{d\theta} = \frac{d}{d\theta}\left[\frac{3i(i+1)}{2(2i-1)(2i+1)}P_{i-2} + \frac{i^2+i-3}{(2i-1)(2i+3)}P_i + \frac{3i(i+1)}{2(2i+1)(2i+3)}P_{i+2}\right], \tag{32}$$

$$P_i\frac{dP_2}{d\theta} = 3\frac{d}{d\theta}\left[\frac{-i}{(2i-1)(2i+1)}P_{i-2} + \frac{1}{(2i-1)(2i+3)}P_i + \frac{i+1}{(2i+1)(2i+3)}P_{i+2}\right], \tag{33}$$

$$\frac{dP_2}{d\theta}\,\frac{d^2P_i}{d\theta^2} = 3\frac{d}{d\theta}\left[\frac{i(i+1)^2}{(2i-1)(2i+1)}P_{i-2} + \frac{i^2+i-3}{(2i-1)(2i+3)}P_i - \frac{i^2(i+1)}{(2i+1)(2i+3)}P_{i+2}\right]. \tag{34}$$

In (32) and (33) we may obviously use $d\mu$ for $d\theta$, where $\mu \equiv \cos\theta$, on the two sides of the equation.

§9. The above results were obtained originally by means of the relations existing between consecutive harmonics, use being made in (34) of the differential equation satisfied by harmonics. The results (30) and (32) may, however, be regarded as particular cases of a formula due to Professor C. Niven.* Employing Heine's notation

$$P_m^n(\mu) = \frac{\underline{|n-m}}{1.3\ldots.(2n-1)}(\mu^2-1)^{\frac{m}{2}}\frac{d^mP_n(\mu)}{d\mu^m},$$

he found

$$\mu^2P_m^n = P_m^{n+2} + \frac{2n^2+2n-2m^2-1}{(2n-1)(2n+3)}P_m^n + \frac{(n^2-m^2)\{(n-1)^2-m^2\}}{(4n^2-1)\{4(n-1)^2-1\}}P_m^{n-2}.$$

Now

$$\mu^2 = \tfrac{1}{3}(2P_2+1),$$

so that we may rewrite this in the form

$$P_2\frac{d^mP_n}{d\mu^m} = \tfrac{3}{2}\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\frac{d^mP_{n+2}}{d\mu^m} + \frac{n(n+1)-3m^2}{(2n-1)(2n+3)}\frac{d^mP_n}{d\mu^n}$$
$$+\tfrac{3}{2}\frac{(n+m)(n+m-1)}{(2n-1)(2n+1)}\frac{d^m}{d\mu^m}P_{n-2}.$$

Writing i for n we deduce (30) by putting $m=0$, and (32) by putting $m=1$.

§10. For facilitating the determination of the arbitrary constants use will also be made of the result that if we have an equation with a *finite* number of terms of the form

$$\Sigma A_i\frac{dP_i}{d\theta} = \Sigma B_i\frac{dP_i}{d\theta}, \tag{35}$$

then

$$B_i = A_i, \text{ etc.}$$

The result is obviously true in its present form if it be true when $d\theta$ is replaced by $d\mu$ where $\mu = \cos\theta$. That it is true in the latter form may easily be proved rigidly by means of the substitution†

$$\frac{dP_i}{d\mu} = (2i-1)P_{i-1} + (2i-5)P_{i-3} + \ldots.$$

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* Phil. Trans. for 1880, p. 124.

† Todhunter's "Functions of Laplace," p. 22.

§11. In each of the identities (30)–(34) there occur on the right only three surface harmonics, P_{i-2}, P_i and P_{i+2}. Thus supposing for simplicity there to be only one surface harmonic P_i in the applied surface forces on the right of equations (27), (28) and (29), we require on the left of these equations in addition to the *principal* terms involving P_i (which would alone occur in a perfect sphere) *subsidiary* terms involving P_i along with the two other harmonics P_{i-2} and P_{i+2}. The arbitrary constants Y_{i-2}, Y_{i+2} .... with suffixes $i-2$ and $i+2$ are only of order ε, so that in the surface values of the subsidiary terms we may suppose $r=a$ simply.

Again in (27), since $\overline{M}_i$ occurs multiplied by ε, we may employ the first approximation given by (28), viz.

$$\overline{M}_i = \Theta_i.$$

Thus supposing only one surface harmonic on the right of (27), we may by means of (30) and (31) write that equation as

$$\begin{aligned}&(A_{i-2}Y_{i-2}+B_{i-2}Z_{i-2})\,P_{i-2}+(L_iV_i+A_iY_i+B_iZ_i)\,P_i\\&+(A_{i+2}Y_{i+2}+B_{i+2}Z_{i+2})\,P_{i+2}+\varepsilon a\frac{d}{da}(L_iV_i+A_iY_i+B_iZ_i)\left\{\frac{3\,(i-1)\,i}{2\,(2i-1)(2i+1)}P_{i-2}\right.\\&\left.+\frac{i\,(i+1)}{(2i-1)(2i+3)}P_i+\frac{3\,(i+1)(i+2)}{2\,(2i+1)(2i+3)}P_{i+2}\right\}\\&-3\varepsilon\Theta_i\left\{\frac{(i-1)\,i\,(i+1)}{(2i-1)(2i+1)}P_{i-2}+\frac{i\,(i+1)}{(2i-1)(2i+3)}P_i-\frac{i\,(i+1)(i+2)}{(2i+1)(2i+3)}P_{i+2}\right\}\\&=R_iP_i\;....\end{aligned}\qquad(36)$$

Equating coefficients of the same zonals on the two sides of (36) we get

$$A_{i-2}Y_{i-2}+B_{i-2}Z_{i-2}=\frac{-3\,(i-1)\,i}{2\,(2i-1)(2i+1)}\,\varepsilon a\frac{d\overline{L}_i}{da}+\frac{3\,(i-1)\,i\,(i+1)}{(2i-1)(2i+1)}\,\varepsilon\Theta_i,\qquad(37)$$

$$A_iY_i+B_iZ_i=R_i-L_iV_i-\frac{i\,(i+1)}{(2i-1)(2i+3)}\,\varepsilon a\frac{d\overline{L}_i}{da}+\frac{3i\,(i+1)}{(2i-1)\,(2i+3)}\,\varepsilon\Theta_i,\qquad(38)$$

$$A_{i+2}Y_{i+2}+B_{i+2}Z_{i+2}=\frac{-3\,(i+1)(i+2)}{2\,(2i+1)(2i+3)}\,\varepsilon a\frac{d\overline{L}_i}{da}-\frac{3i\,(i+1)(i+2)}{(2i+1)\,(2i+3)}\,\varepsilon\Theta_i.\qquad(39)$$

In the second surface condition (28) we make use of (32), (33) and (34), and then, equating the coefficients of $\frac{dP_{i-2}}{d\theta}$, $\frac{dP_i}{d\theta}$ and $\frac{dP_{i+2}}{d\theta}$ respectively on the two sides of the equation, we easily deduce the three equations

$$C_{i-2}Y_{i-2}+D_{i-2}Z_{i-2} = \frac{3\varepsilon i}{(2i-1)(2i+1)}\left\{-\tfrac{1}{2}(i+1)\,a\,\frac{d\overline{M}_i}{da}-\overline{N}_i+(i+1)^2\,\overline{O}_i\right\}, \quad (40)$$

$$C_iY_i+D_iZ_i=\Theta_i-M_iV_i + \frac{\varepsilon}{(2i-1)(2i+3)}\left\{-(i^2+i-3)\,a\,\frac{d\overline{M}_i}{da}+3\overline{N}_i+3\,(i^2+i-3)\,\overline{O}_i\right\}, \quad (41)$$

$$C_{i+2}Y_{i+2}+D_{i+2}Z_{i+2}=\frac{3\varepsilon\,(i+1)}{(2i+1)(2i+3)}\left\{-\tfrac{1}{2}ia\,\frac{d\overline{M}_i}{da}+\overline{N}_i-i^2\overline{O}_i\right\}. \quad (42)$$

The six equations (37)–(42) suffice to determine the six unknowns Y_{i-2}, Z_{i-2}, Y_i, Z_i, Y_{i+2}, Z_{i+2}. Before entering, however, on this determination we will consider the third surface condition (29).

§12. Supposing only the one harmonic P_i in the applied surface forces we require in the third surface condition, as in the first two, subsidiary terms in P_{i-2}, P_i and P_{i+2} alone. Making use of (33) and (34) and equating the coefficients of $\frac{dP_{i-2}}{d\theta}$, $\frac{dP_i}{d\theta}$ and $\frac{dP_{i+2}}{d\theta}$ respectively on the two sides of the equation, we find

$$(i-3)\,a^{i-3}W_{i-2}=3\varepsilon a^{i-1}W_i\,i\,(i+1)(i+2)\div\{(2i-1)(2i+1)\}, \quad (43)$$

$$(i-1)\,a^{i-1}W_i=\frac{1}{n}\Phi_i+9\varepsilon a^{i-1}W_i\,(i-1)(i+2)\div\{(2i-1)(2i+3)\}, \quad (44)$$

$$(i+1)\,a^{i+1}W_{i+2}=-3\varepsilon a^{i-1}W_i\,(i-1)\,i\,(i+1)\div\{(2i+1)(2i+3)\}. \quad (45)$$

From (44), neglecting terms in ε^2, we find

$$W_i=\frac{a^{-i+1}}{i-1}\,\frac{\Phi_i}{n}\left\{1+\frac{9\varepsilon\,(i+2)}{(2i-1)(2i+3)}\right\}, \quad (46)$$

and substituting the principal term in the value of W_i in (43) and (45) we get

$$W_{i-2}=\frac{\Phi_i}{n}\,\frac{3\varepsilon a^{-i+3}\,i\,(i+1)(i+2)}{(i-3)(i-1)(2i-1)(2i+1)}, \quad (47)$$

$$W_{i+2}=-\frac{\Phi_i}{n}\,\frac{3\varepsilon a^{-i-1}\,i}{(2i+1)(2i+3)}. \quad (48)$$

This determines W_{i-2}, W_i and W_{i+2} in terms of the surface force perpendicular to the meridian alone; or there still exists in the present case of symmetry round an axis that independence of the transverse strains and stresses which, as stated in §3, exists in the perfect sphere.

As shown by (46), W_i becomes infinite if $i = 1$, while by (47) W_{i-2} becomes infinite if i be either 1 or 3. The explanation is that in these cases the applied forces are not in equilibrium, so that in addition to elastic displacements rigid body displacements and motion would ensue, and to obtain a complete knowledge of what happens it would be necessary to have recourse to dynamical principles.

§13. We now return to the six equations (37)–(42) supplied by the first and second surface conditions. The quantities $\bar{L}_i$, etc., on the right of these equations contain Y_i and Z_i. For these, since ϵ occurs as a factor, we are to substitute the first approximations given by the equations

$$A_iY_i + B_iZ_i = R_i - L_iV_i,$$
$$C_iY_i + D_iZ_i = \Theta_i - M_iV_i,$$

which occur in the case of the perfect sphere.

Now if for shortness

$$(2i^2 + 4i + 3)\,m - (2i + 1)\,n \equiv \Pi_i, \tag{49}$$

then

$$A_iD_i - B_iC_i = 2n\,(i - 1)\,a^{2i-2}\Pi_i \div \{i\,(i + 1)(2i + 3)\}, \tag{50}$$

and the first approximations are

$$Y_i = (i + 1)\Big[(2i + 3)\,a^{-i}(R_i - i\Theta_i) + \left\{(2i + 3)\,m - n\right\}\frac{\rho}{m + n}\,V_i\Big] \div \Pi_i, \tag{51}$$

$$\begin{aligned} Z_i = \frac{1}{2n}\,\frac{i}{i - 1}\Big[\{i\,(i + 2)\,m - n\}\,a^{-i+2}R_i - (i + 1)\{(i^2 - i - 3)\,m + n\}\,a^{-i+2}\Theta_i \\ + i\,\{(i + 2)\,m - n\}\,\rho a^2V_i\Big] \div \Pi_i\,. \end{aligned} \tag{52}$$

In the right-hand sides of the equations (37)–(42) we first of all carry out differentiations $\frac{d\bar{L}_i}{da}$, etc., as if Y_i and Z_i were constants, and thereafter substitute for these two quantities from (51) and (52). The right-hand sides then contain only known quantities, and we have 3 pairs of simple equations to determine Y_{i-2} and Z_{i-2}, Y_{i+2} and Z_{i+2}, and the secondary terms of Y_i and Z_i. These secondary terms we shall now denote by δY_i and δZ_i. Thus the complete values of Y_i and Z_i as given by (38) and (41) are the sums of the principal terms given in (51) and (52) and the secondary terms to be deduced from

$$A_i\delta Y_i + B_i\delta Z_i = \frac{i(i+1)\varepsilon}{(2i-1)(2i+3)}\left\{-a\frac{d\overline{L}_i}{da} + 3\Theta_i\right\}, \tag{53}$$

$$C_i\delta Y_i + D_i\delta Z_i$$
$$= \frac{\varepsilon}{(2i-1)(2i+3)}\left\{-(i^2+i-3)\,a\frac{d\overline{M}_i}{da} + 3\overline{N}_i + 3(i^2+i-3)\,\overline{O}_i\right\}. \tag{54}$$

In our further discussion of the problem we shall, except when otherwise stated, restrict Y_i and Z_i to mean the first approximations (51) and (52) alone. Thus the Y_i, Z_i occurring in (1)–(4) are equal respectively to $Y_i + \delta Y_i$ and $Z_i + \delta Z_i$, where Y_i and Z_i now denote the first approximations (51) and (52), while δY_i, δZ_i are given by (53) and (54). No such change is required in the meanings to be attached to the other Y and Z constants because there are no principal terms in P_{i-2} or P_{i+2}. If we supposed several harmonics to occur in the applied forces, use might be made of the notation $\delta_s Y_j$ to denote the subsidiary term in the complete value of Y_j arising from the presence of P_s in the applied forces. For instance, we should then regard (37) and (40) as determining $\delta_i Y_{i-2}$ and $\delta_i Z_{i-2}$. The complete value of a Y or Z constant in such a case would be the sum of the first approximation—i. e. the value for a perfect sphere—and of the several subsidiary terms.

§14. From (38) and (41) we see that in the right-hand sides of (37), (39), (40), (42), (53) and (54), since ε is a factor, we may substitute R_i for $\overline{L}_i$ and Θ_i for $\overline{M}_i$. From (6)–(9) and (14)–(17) we see that A_i, C_i, L_i, M_i, N_i and O_i all contain a^i, while B_i and D_i contain a^{i-2}, also that

$$B_i = iD_i. \tag{55}$$

Thus

$$a\frac{d\overline{L}_i}{da} = i(R_i - 2D_iZ_i), \tag{56}$$

$$a\frac{d\overline{M}_i}{da} = i\Theta_i - 2D_iZ_i. \tag{57}$$

By means of (6)–(13), we easily find

$$E_i = \frac{1}{i-1}\{(i^2+i-1)A_i - i(i-2)(i+1)C_i\}, \tag{58}$$

$$F_i = \frac{1}{i-1}B_i, \tag{59}$$

$$G_i = \frac{1}{i-1}\{A_i - (i-3)C_i\}, \tag{60}$$

$$H_i = \frac{1}{i-1}D_i. \tag{61}$$

Thus

$$E_iY_i + F_iZ_i = \frac{1}{i-1}\{(i^2+i-1)(A_iY_i+B_iZ_i)$$

$$-(i-2)\,i\,(i+1)(C_iY_i+D_iZ_i)\} - \frac{2i}{i-1}B_iZ_i$$

$$= \frac{1}{i-1}\{(i^2+i-1)(R_i-L_iV_i) - (i-2)\,i\,(i+1)\,(\Theta_i-M_iV_i)\} - \frac{2i^2}{i-1}D_iZ_i. \quad (62)$$

Similarly

$$G_iY_i + H_iZ_i = \frac{1}{i-1}\{R_i - L_iV_i - (i-3)(\Theta_i - M_iV_i) - 2D_iZ_i\}. \quad (63)$$

Referring to (24), we easily find

$$\overline{N}_i = \frac{1}{i-1}\{i^2\rho a^iV_i + (i^2+i-1)\,R_i - (i-2)\,i\,(i+1)\,\Theta_i - 2i^2D_iZ_i\}, \quad (64)$$

$$\overline{O}_i = \frac{1}{i-1}\{\rho a^iV_i + R_i - (i-3)\,\Theta_i - 2D_iZ_i\}, \quad (65)$$

and, as already pointed out,

$$\overline{L}_i = R_i, \quad (66)$$

$$\overline{M}_i = \Theta_i. \quad (67)$$

§15. Having regard to (56), (57), (64), (65), (66) and (67), we see that Y_i has been got rid of on the right-hand sides of (37), (39), (40), (42), (53) and (54). The equations thus simplified are

$$A_{i-2}Y_{i-2} + B_{i-2}Z_{i-2} = \frac{3\varepsilon i\,(i-1)}{(2i-1)(2i+1)}\{-\tfrac{1}{2}iR_i + (i+1)\,\Theta_i + iD_iZ_i\}, \quad (68)$$

$$C_{i-2}Y_{i-2} + D_{i-2}Z_{i-2} = \frac{3\varepsilon i}{(i-1)(2i-1)(2i+1)}\{(2i+1)\,\rho a^iV_i + (i+2)\,R_i$$

$$-\tfrac{1}{2}(i-3)(i+1)(i+2)\,\Theta_i + (i^2-4i-3)\,D_iZ_i\}, \quad (69)$$

$$A_i\delta Y_i + B_i\delta Z_i = \frac{\varepsilon i\,(i+1)}{(2i-1)(2i+3)}\{-iR_i + 3\Theta_i + 2iD_iZ_i\}, \quad (70)$$

$$C_i\delta Y_i + D_i\delta Z_i = \frac{\varepsilon}{(2i-1)(2i+3)}\{3\,(2i+3)\,\rho a^iV_i + 6\,(i+2)\,R_i$$

$$-(i^3+7i^2-6i-27)\,\Theta_i + 2\,(i^2-5i-12)\,D_iZ_i\}, \quad (71)$$

$$A_{i+2}Y_{i+2} + B_{i+2}Z_{i+2} = -\frac{3\varepsilon i\,(i+1)(i+2)}{2\,(2i+1)(2i+3)}(R_i + 2\Theta_i - 2D_iZ_i), \quad (72)$$

$$C_{i+2}Y_{i+2} + D_{i+2}Z_{i+2} = \frac{3\varepsilon\,(i+1)}{2\,(2i+1)(2i+3)}\{2R_i - i\,(i+4)\,\Theta_i + 2iD_iZ_i\}. \quad (73)$$

In these equations D_i is given by (9) and Z_i by (52), while A_{i-2}, A_{i+2}, etc., are obtained by writing $i - 2$ or $i + 2$, as the case may be, for i in the general formulæ (6), etc.

§16. The algebraical work of solving the equations is not of interest, so it will suffice to state the results. Π_i, it will be remembered, is given by (49), while Π_{i-2}, Π_{i+2} are the corresponding quantities when $i - 2$ and $i + 2$ are written for i. The results are

$$\begin{aligned} Y_{i-2} = \frac{3\varepsilon a^{-i+2} i}{(2i+1)\Pi_{i-2}\Pi_i} \Big[& \{(2i^3 - 4i^2 + 5i + 6)m - (i+2)n\} \rho a^i V_i \\ & - \tfrac{1}{2}\{(2i^5 - 4i^4 - 3i^3 + 16i^2 - 5i - 24)m - (2i^4 + i^3 - 6i^2 - 7i + 4)n\} R_i \\ & + \tfrac{1}{2}(i+1)\{(2i^5 - 6i^4 - i^3 + 21i^2 - 4i + 6)m \\ & - (2i^4 - i^3 - 9i^2 + 12i + 2)n\} \Theta_i \Big], \end{aligned} \tag{74}$$

$$\begin{aligned} Z_{i-2} = \frac{3\varepsilon a^{-i+4}(i-2)i}{2(i-3)(2i-1)(2i+1)n\Pi_{i-2}\Pi_i} \Big[& \{(2i^5 + 4i^4 - 11i^3 + 20i^2 + 3i - 9)m^2 \\ & - i(4i^3 + i^2 + i + 6)mn + (2i+1)(i^2 - i + 1)n^2\} \rho a^i V_i \\ & - \tfrac{1}{2}\{(2i^6 - 10i^5 + 3i^4 + 31i^3 - 50i^2 - 30i + 36)m^2 \\ & - (2i^5 - 3i^4 - 21i^3 + 13i^2 + 3i - 18)mn \\ & + (2i+1)(i^2 - 5i + 2)n^2\} R_i \\ & + \tfrac{1}{2}(i+1)\{i(2i^5 - 16i^4 + 27i^3 + 40i^2 - 56i + 21)m^2 \\ & - (2i^5 - 13i^4 + 16i^3 + 47i^2 - 22i - 6)mn \\ & - (2i+1)(i^2 - 5i + 2)n^2\} \Theta_i \Big], \end{aligned} \tag{75}$$

$$\begin{aligned} \delta Y_i = \frac{\varepsilon a^{-i} i(i+1)}{(2i-1)(\Pi_i)^2} \Big[& 3\{(2i^2 - 2i - 9)m + 3n\} \rho a^i V_i \\ & - \{(2i^4 + 6i^3 + 7i^2 + 21i + 36)m - (2i^3 + 15i^2 + 19i - 12)n\} R_i \\ & + i\{(2i^4 + 6i^3 + i^2 + 9i + 27)m - (2i^3 + 15i^2 + 13i - 15)n\} \Theta_i \Big], \end{aligned} \tag{76}$$

$$\begin{aligned} \delta Z_i = \frac{\varepsilon a^{-i+2} i(i+1)}{2(i-1)(2i-1)(2i+3)n(\Pi_i)^2} \Big[& \{(4i^5 + 22i^4 + 10i^3 - 27i^2 + 45i + 81)m^2 \\ & - (8i^4 + 18i^3 - 5i^2 + 15i + 54)mn + (4i^3 + 2i^2 + 9)n^2\} \rho a^i V_i \\ & - \{(2i^6 + 4i^5 - 5i^4 + 2i^3 + 6i^2 - 90i - 108)m^2 \\ & - i(2i^4 + 15i^3 - 2i^2 - 75i - 66)mn \\ & + (2i^3 - 15i^2 - 20i + 12)n^2\} R_i \\ & + \{(2i^7 - 19i^5 + 12i^4 + 107i^3 + 78i^2 + 9i + 27)m^2 \\ & - (2i^6 + 7i^5 - 12i^4 - 22i^3 + 61i^2 + 90i + 18)mn \\ & - (2i^4 + 11i^3 - 5i^2 - 29i - 3)n^2\} \Theta_i \Big], \end{aligned} \tag{77}$$

$$Y_{i+2} = \frac{-3\varepsilon a^{-i-2}(i+1)(i+2)^2(i+3)(2i+7)}{2(2i+1)(2i+3)\Pi_{i+2}} (R_i - i\Theta_i), \tag{78}$$

$$\begin{aligned} Z_{i+2} = \frac{3\varepsilon a^{-i}(i+2)}{4(2i+1)(2i+3)n\Pi_i\Pi_{i+2}} & \left[2i^2\{(i+2)m-n\}\Pi_{i+2}\rho a^i V_i \right. \\ & -\{(2i^6+20i^5+75i^4+136i^3+136i^2+72i-18)m^2 \\ & -(2i^5+19i^4+56i^3+43i^2-42i-24)mn \\ & +(2i^3-3i^2-22i-6)n^2\}R_i \\ & +i\{(2i^6+16i^5+43i^4+44i^3+24i^2+15i-18)m^2 \\ & -(2i^5+11i^4+12i^3-33i^2-76i-24)mn \\ & \left. -(2i^3+15i^2+27i+6)n^2\}\Theta_i\right]. \end{aligned} \tag{79}$$

§17. Referring to (2), (3) and (4), we are now in possession of the complete solution of the problem. The i^{th} surface harmonic involved is P_i. Of the constant coefficients Y_i is the sum of (51) and (76), Z_i the sum of (52) and (77), while W_i is given by (46). The $(i-2)^{\text{th}}$ harmonic is P_{i-2}, and of the constant coefficients Y_{i-2} is given by (74), Z_{i-2} by (75) and W_{i-2} by (47). The $(i+2)^{\text{th}}$ harmonic is P_{i+2}, and Y_{i+2} is given by (78), Z_{i+2} by (79) and W_{i+2} by (48). The potential of the bodily forces is

$$r^i V_i P_i,$$

so that V_i is given, while V_{i-2} and V_{i+2} are zero.

After this explanation it seems unnecessary to record the values of the displacements explicitly.

The expressions for the arbitrary constants are long; but in any particular case, as i and m/n are then given numerical quantities, they become easily manageable. It must also be borne in mind that we have here the solution of a great variety of problems, and that in most actual cases only one or two of the types of applied forces will exist.

§18. The previous method is equally applicable to the case when the applied forces involve P_2 only while the equation to the surface is

$$r = a(1 + \Sigma\varepsilon_i P_i),$$

where the squares and products of the constants represented by ε_i are negligible.

For brevity we may suppose but the one harmonic P_i in the equation to the surface.

Taking for instance the first and second surface conditions, we have for these

$$\widehat{rr} - \varepsilon_i \frac{dP_i}{d\theta}\widehat{r\theta} = R_2 P_2,$$

$$\widehat{r\theta} - \varepsilon \frac{dP_i}{d\theta}\widehat{\theta\theta} = \Theta \frac{dP_2}{d\theta},$$

where R_2 and Θ_2 are constants.

In the first surface condition we have principal terms in P_2 and subsidiary from P_2P_i and $\frac{dP_i}{d\theta}\frac{dP_2}{d\theta}$, i. e. in P_{i-2}, P_i and P_{i+2}. In the second surface condition the principal terms are in $\frac{dP_2}{d\theta}$; the subsidiary come from

$$P_2\frac{dP_i}{d\theta},\ P_i\frac{dP_2}{d\theta} \text{ and } \frac{dP_i}{d\theta}\frac{d^2P_2}{d\theta^2}.$$

But
$$\frac{dP_i}{d\theta}\frac{d^2P_2}{d\theta^2}=\frac{dP_i}{d\theta}-4P_2\frac{dP_i}{d\theta},$$

and so using the results (32) and (33), we see the subsidiary terms are in $\frac{dP_{i-2}}{d\theta}$, $\frac{dP_i}{d\theta}$, $\frac{dP_{i+2}}{d\theta}$.

An examination of the third surface condition leads to similar results. The problem is thus obviously soluble in precisely the same way as the preceding.

§19. The method may be applied to most, if not all, cases where over a surface

$$r=a(1+\varepsilon_jP_j),$$

where ε_j^2 is negligible, there act along r, θ, ϕ respectively

$$R_iP_i,\ \Theta_i\frac{dP_i}{d\theta},\ -\Phi_i\frac{dP_i}{d\theta},$$

where R_i, Θ_i, Φ_i represent constants.

Supposing P_iP_j expressed in terms of P_{i+j}, P_{i+j-2}, etc., $P_i\frac{dP_j}{d\theta}$ in terms of $\frac{d}{d\theta}P_{i+j}$, etc., and so on, there is no theoretical difficulty in applying the solution so as to deduce the values of the subsidiary constants represented by δ_jY_i, denoting the change in Y_i due to the presence of P_j in the equation to the surface. As i and j increase, however, the complexity of the solution rapidly increases. For instance, P_4P_i when expanded contains five harmonics, viz. P_{i-4}, P_{i-2}, P_i, P_{i+2} and P_{i+4}.

SECTION II.

§20. In our first section, surface forces were supposed resolved along r and θ. It seems, however, likely to be more convenient in some cases to have them resolved along the normal and in the tangent plane. As in the final problem of Section I we shall confine our attention to a spheroid of very small eccentricity,

$$r = a(1 + \varepsilon P_2),$$

under forces symmetrical about the axis of figure.

If N and T represent the normal and tangential forces—the latter being supposed to act in the meridian, as the component perpendicular to the meridian would be treated as in Section I—then we are given

$$N = \Sigma N_i P_i, \tag{1}$$

$$T = \Sigma T_i \frac{dP_i}{d\theta}. \tag{2}$$

Taking the i^{th} harmonic as a type, we may regard these forces as equivalent to forces R, Θ along r and θ, where

$$R = N_i P_i + \varepsilon \frac{dP_2}{d\theta} T_i \frac{dP_i}{d\theta}, \tag{3}$$

$$\Theta = T_i \frac{dP_i}{d\theta} - \varepsilon \frac{dP_2}{d\theta} N_i P_i, \tag{4}$$

ε^2 being neglected as before.

Referring to (31) and (33), Section I, we can express $\frac{dP_2}{d\theta}\frac{dP_i}{d\theta}$ in terms of P_{i-2}, P_i and P_{i+2}, and can express $P_i \frac{dP_2}{d\theta}$ in terms of the differentials of the same three harmonics.

Thus the first and second surface conditions may be written

$$\widehat{rr} - \varepsilon \frac{dP_2}{d\theta}\widehat{r\theta} = N_i P_i + 3\varepsilon T_i i(i+1)\left\{\frac{i-1}{(2i-1)(2i+1)} P_{i-2} \right.$$
$$\left. + \frac{1}{(2i-1)(2i+3)} P_i - \frac{i+2}{(2i+1)(2i+3)} P_{i+2}\right\}, \tag{5}$$

$$\widehat{r\theta} - \varepsilon \frac{dP_2}{d\theta}\widehat{\theta\theta} = T_i \frac{dP_i}{d\theta} - 3\varepsilon N_i \frac{d}{d\theta}\left\{-\frac{i}{(2i-1)(2i+1)} P_{i-2} \right.$$
$$\left. + \frac{1}{(2i-1)(2i+3)} P_i + \frac{i+1}{(2i+1)(2i+3)} P_{i+2}\right\}. \tag{6}$$

It is obvious that we require only the terms depending on P_{i-2}, P_i and P_{i+2} in the general solution (2), (3), (4) of Section I. The method of procedure is also obviously identical with that of Section I, so that it will suffice to record the values of Y_{i-2}, Z_{i-2}, δY_i, δZ_i, Y_{i+2} and Z_{i+2}. The first approximations (51) and (52), Section I, for Y_i and Z_i are, it is needless to say, applicable in the present case. With the notation (49) of Section I we now get

$$\begin{aligned} Y_{i-2} = \frac{3\varepsilon a^{-i+2} i}{2(2i+1)\Pi_{i-2}\Pi_i} \Big[& -\{(2i^5-7i^3+6i^2-7i-12)m \\ & -(2i^4+5i^3-16i^2-5i+8)n\}N_i \\ & +(i+1)\{(2i^5-2i^4-i^3+15i^2-8i+12)m \\ & -(2i^4+3i^3-15i^2+12i+4)n\}T_i\Big], \end{aligned} \tag{7}$$

$$\begin{aligned} Z_{i-2} = \frac{3\varepsilon a^{-i+4}(i-2)i}{4(i-3)(2i-1)(2i+1)\Pi_{i-2}\Pi_i} \times & \\ \Big[& -\{(2i^6-6i^5-13i^4+21i^3-34i^2-6i+18)m^2 \\ & -(2i^5+i^4-47i^3+29i^2+9i-18)mn+(2i+1)(i^2-7i+4)n^2\}N_i \\ & +(i+1)\{i(2i^5-12i^4+23i^3+30i^2-58i+33)m^2 \\ & -(2i^5-9i^4+10i^3+53i^2-20i-12)mn \\ & -(2i+1)(i^2-7i+4)n^2\}T_i\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} \delta Y_i = \frac{\varepsilon a^{-i} i(i+1)}{(2i-1)(\Pi_i)^2} \Big[& -\{(2i^4+6i^3+i^2+9i+27)m-(2i^3+15i^2+13i-15)n\}N_i \\ & +\{(2i^5+6i^4+7i^3+27i^2+48i+9)m \\ & -(2i^4+15i^3+19i^2-6i+3)n\}T_i\Big], \end{aligned} \tag{9}$$

$$\begin{aligned} \delta Z_i = \frac{\varepsilon a^{-i+2} i(i+1)}{2n(i-1)(2i-1)(2i+3)(\Pi_i)^2} \times & \\ \Big[& -\{(2i^6+4i^5-11i^4-4i^3+27i^2-45i-81)m^2 \\ & -(2i^5+15i^4-8i^3-66i^2-33i+18)mn \\ & +(2i^3-15i^2-14i+15)n^2\}N_i \\ & +\{(2i^7-13i^5+36i^4+140i^3+96i^2+9i+27)m^2 \\ & -(2i^6+7i^5-6i^4-i^3+79i^2+99i+18)mn \\ & -(2i^4+11i^3-11i^2-32i-3)n^2\}T_i\Big], \end{aligned} \tag{10}$$

$$Y_{i+2} = \frac{3\varepsilon a^{-i-2} i(i+1)(i+2)(i+3)(2i+7)}{2(2i+1)(2i+3)\Pi_{i+2}}(iT_i-N_i), \tag{11}$$

$$\begin{aligned} Z_{i+2} = \frac{3\varepsilon a^{-i} i(i+2)}{4(2i+1)(2i+3)n\Pi_i\Pi_{i+2}} \Big[& -\{(i+2)(2i^4+12i^3+19i^2+12i+24)m^2 \\ & -(2i^4+15i^3+34i^2+19i-16)mn+(2i^2+i-8)n^2\}N_i \\ & +\{(2i^6+12i^5+3i^4-106i^3-248i^2-233i-114)m^2 \\ & -(2i^5+7i^4-26i^3-145i^2-202i-68)mn \\ & -(2i^3+19i^2+37i+10)n^2\}T_i\Big]. \end{aligned} \tag{12}$$

For the values of the displacements we employ as before (2), (3) and (4) of Section I. In these we substitute the above values of Y_{i-2}, Z_{i-2}, Y_{i+2} and Z_{i+2}.

The complete values of Y_i and Z_i are the sums of (9) and (51), Section I, and of (10) and (52), Section I, respectively.

The expressions for the arbitrary constants are, in point of length, pretty much on a par with those of the first problem.

As the opportunity for numerical slips in such problems as the present is considerable, I may mention that after working out the present problem and that of Section I independently, I employed the one solution to verify the other.

§21. Whether the surface forces be resolved along the radius and its perpendicular or along the normal and tangent, the case $i=2$ presents special features. The harmonic P_0 occurring in the subsidiary terms is simply unity, so that all terms containing it are independent of the angular coordinates and answer to pure radial displacements.

The values (75), Section I, and (8) vanish when we write 2 for i, and this they ought to do, for in the general solution there is only one pure radial displacement term, and that answers to Y_0.

Besides presenting these peculiarities, the case $i=2$ seems of special physical interest. I shall thus briefly consider the problem presented by the nearly spherical spheroid

$$r=a(1+\varepsilon P_2), \tag{13}$$

acted on by the surface forces

$$\left.\begin{aligned} N_2P_2 &\equiv \tfrac{1}{2}(3\cos^2\theta-1)N_2 \text{ along the normal,}\\ T_2\frac{dP_2}{d\theta} &\equiv -3\sin\theta\cos\theta T_2 \quad \text{`` `` meridian tangent,}\\ -\Phi_2\frac{dP_2}{d\theta} &\equiv \quad 3\sin\theta\cos\theta\Phi_2 \quad \text{`` `` perpendicular to the meridian.} \end{aligned}\right\}(14)$$

From (7)–(12), and from (51), (52), and (46)–(48) of Section I, we have

$$Y_0=\frac{18\varepsilon\{-(m-n)N_2+4(10m-3n)T_2\}}{5(3m-n)(19m-5n)}, \tag{15}$$

$$Y_2=\frac{21a^{-2}(N_2-2T_2)}{19m-5n}+\frac{6\varepsilon a^{-2}\{-(43m-29n)N_2+(143m-73n)T_2\}}{(19m-5n)^2}, \tag{16}$$

$$\begin{aligned} Z_2=&\frac{(8m-n)N_2+3(m-n)T_2}{n(19m-5n)}\\ &\quad+\frac{3}{7}\varepsilon\frac{(m-n)(5m-19n)N_2+(655m^2-260mn-3n^2)T_2}{n(19m-5n)^2}, \end{aligned} \tag{17}$$

$$Y_4=\frac{132}{7}\,\frac{\varepsilon a^{-4}(2T_2-N_2)}{17m-3n}, \tag{18}$$

$$Z_4 = -\frac{4\varepsilon a^{-2}}{35}\,\frac{(504m^2 - 155mn + n^2)\,N_2 + 2\,(465m^2 - 271mn + 44n^2)\,T_2}{n\,(19m - 5n)(17m - 3n)}, \tag{19}$$

$$W_2 = a^{-1}\frac{\Phi_2}{n}\left(1 + \frac{12}{7}\,\varepsilon\right), \tag{20}$$

$$W_4 = -\frac{6}{35}\,\varepsilon a^{-3}\,\frac{\Phi_2}{n}. \tag{21}$$

As already stated Z_0 vanishes, and $\frac{dP_0}{d\theta}$ being zero, there is no term in W_0. Under Y_2 and Z_2 above are included both principal and subsidiary terms.

Substituting in (2), (3) and (4) of Section I, we find for the displacements:

$$\begin{aligned} u = &-\frac{6}{5}\,\varepsilon r\,\frac{(m-n)\,N_2 - 4\,(10m-3n)\,T_2}{(3m-n)(19m-5n)} \\ &+ \frac{N_2P_2}{n\,(19m-5n)}\biggl[r\left\{8m - n + \frac{3}{7}\,\varepsilon\,\frac{(m-n)(5m-19n)}{19m-5n}\right\} \\ &- 3a^{-2}r^3\,(m-n)\left\{1 - \frac{2}{7}\,\varepsilon\,\frac{43m-29n}{19m-5n}\right\}\biggr] \\ &+ \frac{3\,T_2P_2}{n\,(19m-5n)}\biggl[r\left\{m - n + \frac{\varepsilon}{7}\,\frac{655m^2 - 260mn - 3n^2}{19m-5n}\right\} \\ &+ 2a^{-2}r^3\,(m-n)\left\{1 - \frac{\varepsilon}{7}\,\frac{143m-73n}{19m-5n}\right\}\biggr] \\ &- \frac{4\varepsilon N_2P_4}{35n\,(17m-3n)}\left\{a^{-2}r^3\,\frac{504m^2 - 155mn + n^2}{19m-5n} - 15a^{-4}r^5\,(2m-n)\right\} \\ &- \frac{8\varepsilon T_2P_4}{35n\,(17m-3n)}\left\{a^{-2}r^3\,\frac{465m^2 - 271mn + 44n^2}{19m-5n} + 15a^{-4}r^5\,(2m-n)\right\}, \end{aligned} \tag{22}$$

$$\begin{aligned} v = &\,\frac{1}{2n\,(19m-5n)}\,N_2\,\frac{dP_2}{d\theta}\biggl[r\left\{8m - n + \frac{3}{7}\,\varepsilon\,\frac{(m-n)(5m-19n)}{19m-5n}\right\} \\ &- a^{-2}r^3\,(5m+2n)\left\{1 - \frac{2}{7}\,\varepsilon\,\frac{43m-29n}{19m-5n}\right\}\biggr] \\ &+ \frac{1}{n\,(19m-5n)}\,T_2\,\frac{dP_2}{d\theta}\biggl[\frac{3}{2}\,r\left\{m - n + \frac{\varepsilon}{7}\,\frac{655m^2 - 260mn - 3n^2}{19m-5n}\right\} \\ &+ a^{-2}r^3\,(5m+2n)\left\{1 - \frac{\varepsilon}{7}\,\frac{143m-73n}{19m-5n}\right\}\biggr] \\ &- \frac{\varepsilon N_2}{35n\,(17m-3n)}\,\frac{dP_4}{d\theta}\left\{a^{-2}r^3\,\frac{504m^2 - 155mn + n^2}{19m-5n} - 6a^{-4}r^5(7m+2n)\right\} \\ &- \frac{2\varepsilon T_2}{35n(17m-3n)}\,\frac{dP_4}{d\theta}\left\{a^{-2}r^3\,\frac{465m^2 - 271mn + 44n^2}{19m-5n} + 6a^{-4}r^5(7m+2n)\right\}, \end{aligned} \tag{23}$$

$$w = -a^{-1}\,\frac{\Phi_2}{n}\,\frac{d}{d\theta}\left[\left(1 + \frac{12}{7}\,\varepsilon\right)r^2P_2 - \frac{6}{35}\,\varepsilon a^{-2}r^4P_4\right]. \tag{24}$$

§22. An important question as regards the interpretation of such solutions as the present claims a passing notice. The question is whether $r = a(1 + \epsilon P_2)$ is to be regarded as the equation to the surface before or after the application of the forces. I have elsewhere referred to this subject, but shall treat it independently here.

According to the usual definition $\frac{du}{dr}$, taken as a typical strain, is the limit of $\delta u/\delta r$ when δr is indefinitely small, δr being the original length of an element the increment in which, due to applied force, is δu. According to this definition r, θ, ϕ must apparently refer to the original unstrained condition. The surface forces, however, balance the stresses over, not the original surface before strain, but the surface which answers to the equilibrium position. Further, the bodily forces at least, when such exist, must depend on the positions of the elements of the solid in the strained state.

The simplest way of surmounting the difficulty is to have recourse to the fact that the strains must be very small in order that the mathematical theory which assumes them superposable may apply, and the same limitation is also in general imposed by the narrowness of the limits experiment assigns to *linear** elasticity. Thus when regard is had to the limits imposed either by the mathematical or physical necessities, it is in general immaterial whether the dr in $\frac{du}{dr}$ refers to the strained or the unstrained body. On the other hand, the surface conditions must seemingly refer to the actual equilibrium surface and so to the strained state.

The conclusion appears to be that we may in general suppose r, θ, ϕ to refers throughout to the strained condition. In cases where the limits of linear elasticity are unusually great, this conclusion should, however, not be acted on without careful consideration.

In problems such as the present, where the difference between the strained and unstrained surfaces is necessarily very small, it seems probable that the two methods of regarding r, θ, ϕ will lead to results which in general differ only by terms containing the square or a higher power of the strains, quantities whose retention in a mathematical theory that supposes strains superposable is extremely risky.

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* See Todhunter and Pearson's "History," footnote, p. 9, Vol. II, Part I.

§23. The preceding remarks will show that the determination of the change in form of the surface requires careful treatment. If r, θ, ϕ refer to the strained position, the radius vector to an element on the surface (13) has been both altered in length and rotated. The increment in length is u, the rotation* v/r, where u, v, r have their surface values. Let us take for surface values

$$u = a\,\{A\varepsilon + (B + C\varepsilon)\,P_2 + D\varepsilon P_4\},$$
$$v = aE\frac{dP_2}{d\theta} + \text{terms in } \varepsilon,$$

where A, B, etc., may be easily deduced from (22) and (23). As terms containing ε^2 have been neglected in u, the original surface must have had for its equation

$$r' = a'\,(1 + \varepsilon' P_2 + \varepsilon'' P_4),$$

where

$$\begin{aligned} a\,(1 + \varepsilon P_2) - a\,\{A\varepsilon + (B + C\varepsilon)\,P_2 + D\varepsilon P_4\} \\ = a'\left\{1 + \frac{\varepsilon'}{2}\left(3\cos^2\left(\theta - E\frac{dP_2}{d\theta}\right) - 1\right) + \varepsilon'' P_4\right\}, \\ = a'\left\{1 + \varepsilon' P_2 - \varepsilon' E\left(\frac{6}{5} + \frac{6}{7}P_2 - \frac{72}{35}P_4\right) + \varepsilon'' P_4\right\}. \end{aligned} \tag{25}$$

In obtaining this, since ε' and ε are of the same order, except when one of the two is vanishingly small, we neglect terms in ε and write a for r in the coefficient of ε'. The variation of θ in the term $\varepsilon'' P_4$ is also neglected.

Equating the constant terms and the coefficients of P_2 and P_4 on the two sides of (25), we obtain

$$a\,(1 - A\varepsilon) = a'\left(1 - \frac{6}{5}E\varepsilon'\right), \tag{26}$$

$$a\,(\varepsilon - B - C\varepsilon) = a'\varepsilon'\left(1 - \frac{6}{7}E\right), \tag{27}$$

$$-aD\varepsilon = a'\left(\varepsilon'' + \frac{72}{35}E\varepsilon'\right). \tag{28}$$

Referring to (22) and (23), it will be seen that the constants A, B, C, D, E are of the same order of magnitude as the principal strains in the solid. It may thus appear doubtful what reliance can be placed on the subsidiary terms in $A\varepsilon$, $E\varepsilon'$, etc. The following argument, for instance, might be advanced: "The mathematical theory assumes strains superposable, which is practically equivalent to neglecting their squares, and the previous work is based on the hypothesis that ε^2 is negligible. It would also appear from the above equations that ε' is of the same order of magnitude as $\varepsilon - B$. Thus whether ε be great or small compared to the principal strains, or be of the same order of magnitude, the subsidiary terms, such as $A\varepsilon$, $E\varepsilon'$, in the above equations are not quantities that can be legitimately retained, and we cannot go beyond the approximations

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* The rotation answering to w may be neglected.

$$a' = a,$$
$$\varepsilon' = \varepsilon - B."$$

This argument is not, however, I believe, altogether sound. Our assumption is not that terms in ε^2 are, absolutely considered, negligible, but that terms in ε^2 may be neglected compared to terms in ε when both exist with numerical coefficients of similar magnitudes. What equation (26) shows, for instance, is that

$$1 - a'/a = \varepsilon \times \text{quantity of order of a strain},$$

and we know that while the right-hand side may be small compared to ε^2, yet it is large compared to any term containing ε^2 that has been neglected, because any such term would also have for numerical coefficient a quantity of the order of a strain. Thus the relation (26) can probably be relied on except in an exceptional case. With regard to the equation (28) and the secondary terms in (27) I should hesitate to express an opinion without actually carrying the solution as far as terms in ε^2. Considering the extreme smallness of the quantities involved, the labor required for this purpose appears excessive compared to the probable value of the results. For practical purposes the result

$$\varepsilon = \varepsilon' + B$$

is probably in general sufficiently exact.

If, for convenience, we write

$$(m - n)/2m = \eta,$$

so that η denotes Poisson's ratio, we find, referring to (22) for the value of B, that this becomes

$$\varepsilon = \varepsilon' + \{(7 - 4\eta)N_2 + 18\eta T_2\} \div \{2n(7 + 5\eta)\}. \qquad (29)$$

The displacement v in the meridian plane makes, it will be noticed, no appreciable contribution to the change in the form of the surface. This change may thus be deduced from the radial displacement alone.

§24. If in (29) we put $\varepsilon' = 0$ we simply obtain the value we should have got for ε from the ordinary solution for the perfect sphere, i. e. the solution which treats $r = a$ as the surface over which the forces act.

If in (29)

$$N_2 = 2T_2, \tag{30}$$

i. e. if the surface forces consist of

a normal force $T_2(3\cos^2\theta - 1)$,	outwards	inwards
and a tangential meridian force $T_2(3\sin\theta\cos\theta)$,	towards the poles,	towards the equator,

then

$$\varepsilon = \varepsilon' + T_2/n, \tag{31}$$

or the change in eccentricity is independent of the value of Poisson's ratio.

§25. The existence in the value of u, as given by (22), of terms independent of the angular coordinates necessitates the existence of corresponding constant terms in the dilatation. We have in fact

$$\Delta = -\frac{18}{5}\varepsilon\frac{(m-n)N_2 - 4(10m-3n)T_2}{(3m-n)(19m-5n)} + \frac{21a^{-2}r^2P_2(N_2 - 2T_2)}{19m-5n}$$
$$+ \text{ terms in } \varepsilon P_2 \text{ and } \varepsilon P_4. \tag{32}$$

These constant terms are of importance from their influence on the increment $\delta\bar{v}$ in the volume $\bar{v}$ of the spheroid. This is given by

$$\delta\bar{v} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a(1+\varepsilon P_2)} \Delta r^2 \sin\theta d\phi d\theta dr.$$

Since terms in ε^2 are neglected, the terms in εP_2 and εP_4 in the value of Δ contribute nothing to $\delta\bar{v}$. After a little reduction we easily find

$$\delta v/\bar{v} = \frac{3}{5}\varepsilon(N_2 + 2T_2)/k, \tag{33}$$

where

$$k \equiv m - n/3$$

is the bulk modulus.

The existence of a change of volume forms an important distinction between the spheroid and a sphere under surface forces depending on a harmonic.

The formula (33) affords us the means of testing the accuracy of our solution so far as regards the constant terms in Δ and the terms independent of the angular coordinates in u. For the change of volume produced by any system of surface forces, whose components parallel to rectangular axes are represented by F, G, H per unit of surface S, is given* by

$$3k\delta\bar{v} = \int\int (Fx + Gy + Hz)\,dS,$$

where the integral is taken over the entire surface. In polar coordinates this becomes

$$3k\delta\bar{v} = \int\int r \text{ (radial component of surface forces) } dS.$$

Now in the present case, terms in ε^2 being neglected,

$$r = a(1 + \varepsilon P_2),$$
$$dS = a^2 (1 + 2\varepsilon P_2) \sin\theta\, d\theta\, d\phi,$$

radial component $= N_2 P_2 + \varepsilon T_2 \left(\frac{dP_2}{d\theta}\right)^2$.

Thus we have

$$3k\delta\bar{v} = \varepsilon a^3 \int_0^{\pi}\int_0^{2\pi} \left\{ 3N_2 P_2^2 + T_2 \left(\frac{dP_2}{d\theta}\right)^2 \right\} \sin\theta\, d\theta\, d\phi,$$
$$= \varepsilon a^3 \pi \left(\frac{12}{5} N_2 + \frac{24}{5} T_2\right),$$

which agrees with (33).

Combining (29) and (33), we see that in the case of a nearly spherical spheroid whose unstrained surface is a true sphere

$$\text{if } T_2 \text{ vanish} \begin{cases} \varepsilon \text{ is of the same sign as } N_2, \\ \delta\bar{v} \text{ `` `` `` `` } \varepsilon N_2, \end{cases}$$
$$\text{if } N_2 \text{ vanish} \begin{cases} \varepsilon \text{ `` `` `` `` } T_2, \\ \delta\bar{v} \text{ `` `` `` `` } \varepsilon T_2. \end{cases}$$

It follows that if either normal forces $N_2 P_2$ alone, or tangential forces $T_2 \frac{dP_2}{d\theta}$ alone, tend to hold an originally spherical body in the shape of a spheroid, the volume of the spheroid is necessarily greater than that of the sphere. If normal and tangential forces coexist, the spheroid may be less in volume than the sphere. This will happen, for instance, if $\eta = 1/4$, while N_2 and T_2 are of opposite signs and so related that

$$\frac{4}{3} > \left(\frac{-T_2}{N_2}\right) > \frac{1}{2}. \qquad (34)$$

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* See Camb. Phil. Soc. Trans., Vol. XV, p. 318.

SECTION III.

§26. When the surface of the solid, though nearly spherical, is not symmetrical round an axis, or when there is an absence of such symmetry in the applied forces, the same method may still be employed, but the increased number of subsidiary terms enhances the complexity. I shall treat only one such class of problem here. Before entering on it the notation requires explanation. Denoting the i^{th} zonal harmonic by P_i, I put

$$P_{i,\,k}(\mu) = \frac{\lfloor\!\underline{\,i-k\,}}{1.3\,\ldots\,.\,(2i-1)}(1-\mu^2)^{\frac{k}{2}}\frac{d^k}{d\mu^k}P_i(\mu), \tag{1}$$

where $\mu \equiv \cos\theta$.

The only change from Heine's notation is that the i is lowered so that it may not appear a power. The μ need not in general be written in.

Taking the ordinary coordinates r, θ, ϕ, let

$$\left.\begin{aligned}
\cos(\phi-\alpha_1)\sin\theta\cos\theta &\equiv \cos(\phi-\alpha_1)\,P_{2,\,1} = X_{2,\,1},\\
\cos(\phi-\beta_1)\sin\theta\cos\theta &\equiv \cos(\phi-\beta_1)\,P_{2,\,1} = X'_{2,\,1},\\
\cos(2\phi-\alpha_2)\sin^2\theta &\equiv \cos(2\phi-\alpha_2)\,P_{2,\,2} = X_{2,\,2},\\
\cos(2\phi-\beta_2)\sin^2\theta &\equiv \cos(2\phi-\beta_2)\,P_{2,\,2} = X'_{2,\,2}.
\end{aligned}\right\} \tag{2}$$

Then I propose to find the solution of the elastic solid problem presented by the nearly spherical ellipsoid

$$r = a\,(1+\varepsilon P_2+\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2}) \tag{3}$$

under the action of the bodily forces derived from the potential

$$r^2\,(V_2 P_2 + V_{2,\,1}X'_{2,\,1} + V_{2,\,2}X'_{2,\,2}), \tag{4}$$

combined with the surface forces of which

$$\left.\begin{aligned}
&\text{component along } r = R_2P_2 + R_{2,\,1}X'_{2,\,1} + R_{2,\,2}X'_{2,\,2},\\
&\quad\text{"}\qquad\text{"}\qquad \theta = \frac{d}{d\theta}(\Theta_2P_2+\Theta_{2,\,1}X'_{2,\,1}+\Theta_{2,\,2}X'_{2,\,2})\\
&\qquad\qquad\qquad\qquad + \frac{1}{\sin\theta}\frac{d}{d\phi}(\Phi_{2,\,1}X'_{2,\,1}+\Phi_{2,\,2}X'_{2,\,2}),\\
&\quad\text{"}\qquad\text{"}\qquad \phi = \frac{1}{\sin\theta}\frac{d}{d\phi}(\Theta_{2,\,1}X'_{2,\,1}+\Theta_{2,\,2}X'_{2,\,2})\\
&\qquad\qquad\qquad\qquad - \frac{d}{d\theta}(\Phi_2P_2+\Phi_{2,\,1}X'_{2,\,1}+\Phi_{2,\,2}X'_{2,\,2}).
\end{aligned}\right\} \tag{5}$$

Here ε, ε_1, ε_2 are small positive or negative quantities whose squares and products are negligible; V, R, Θ, Φ denote constants. To economise letters, $X'_{2,1}$ and $X'_{2,2}$ are each employed in four classes of applied forces. For instance, $X'_{2,1}$ appears in the bodily forces and radial surface forces as well as in the surface forces derived from a potential and those derived from a stream function. There is in consequence an apparent loss of generality, the expressions for all these forces seeming to contain the same constant β_1. In reality, however, we can tell by the coefficient appearing in the final solution, according as it is V, R, Θ or Φ, from which class of forces each single term comes, so that the loss of generality is only apparent.

§27. It may, if necessary, be assumed that the applied forces as a whole are in equilibrium. It would, however, seemingly be a mistake in any case to use the resulting equations at this stage to eliminate anything. Each set of terms which, equated to zero, embodies one of the equations of equilibrium constitutes, as it were, an insoluble nucleus which can be isolated at different stages of the proceedings and affords a useful check on the accuracy of the calculations. Further, when a solution is found which takes account of the strains and stresses induced by the motion, when the applied forces are not in equilibrium, the restriction of equilibrium need not be applied, so that at present it seems expedient not to introduce this restriction sooner than avoidable.

In passing we may glance at a difficulty which may arise when unequilibrated forces exist. Take for example a sphere acted on by a constant couple round an axis; the rate of rotation continually increases, and the same would be true of the mathematical expressions for the strains answering to the "centrifugal" forces. It would, however, be illegitimate to treat the results as having any physical application after a certain time has elapsed, for the solid would in due course fly into pieces. A similar result would certainly follow in many other cases of unequilibrated forces. There are cases, however, when rupture is not a necessary consequence of the absence of equilibrium. The applied forces, for instance, may go through a regular cycle, the time integral throughout a cycle of a component in any particular direction vanishing. If the periodic time is small, the dynamical features of the case predominate, but when the change in the forces is very slow, it is otherwise.

§28. A number of pure mathematical results are employed in the solution of the problem. The following are merely special forms of the $P_{i,k}$ of (1):

$$\left.\begin{aligned}
P_{1,1} &= \sin\theta\frac{dP_1}{d\mu} = \sin\theta,\\
P_{2,1} &= \frac{1}{3}\sin\theta\frac{dP_2}{d\mu} = \sin\theta\cos\theta,\\
P_{2,2} &= \frac{1}{3}\sin^2\theta\frac{d^2P_2}{d\mu^2} = \sin^2\theta,\\
P_{3,1} &= \frac{2}{15}\sin\theta\frac{dP_3}{d\mu} = \frac{1}{5}\sin\theta\,(5\cos^2\theta - 1),\\
P_{3,2} &= \frac{1}{15}\sin^2\theta\frac{d^2P_3}{d\mu^2} = \sin^2\theta\cos\theta,\\
P_{3,3} &= \frac{1}{15}\sin^3\theta\frac{d^3P_3}{d\mu^3} = \sin^3\theta,\\
P_{4,1} &= \frac{2}{35}\sin\theta\frac{dP_4}{d\mu} = \frac{1}{7}\sin\theta\cos\theta\,(7\cos^2\theta - 3),\\
P_{4,2} &= \frac{2}{105}\sin^2\theta\frac{d^2P_4}{d\mu^2} = \frac{1}{7}\sin^2\theta\,(7\cos^2\theta - 1),\\
P_{4,3} &= \frac{1}{105}\sin^3\theta\frac{d^3P_4}{d\mu^3} = \sin^3\theta\cos\theta,\\
P_{4,4} &= \frac{1}{105}\sin^4\theta\frac{d^4P_4}{d\mu^4} = \sin^4\theta.
\end{aligned}\right\}\qquad(6)$$

§29. The next set of results are employed in the surface conditions in dealing with those terms that refer to bodily forces, radial surface forces, and surface forces with Θ coefficients, i. e. which come from a potential. Some are probably old. The several X's are defined by (2). The results are

$$(P_2)^2 = \frac{1}{5} + \frac{2}{7}P_2 + \frac{18}{35}P_4, \qquad(7)$$

$$\left(\frac{dP_2}{d\theta}\right)^2 = \frac{6}{5} + \frac{6}{7}P_2 - \frac{72}{35}P_4, \qquad(8)$$

$$P_2\frac{dP_2}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{7}P_2 + \frac{9}{35}P_4\right), \qquad(9)$$

$$\frac{dP_2}{d\theta}\,\frac{d^2P_2}{d\theta^2} = \frac{d}{d\theta}\left(\frac{3}{7}P_2 - \frac{36}{35}P_4\right), \qquad(10)$$

$$P_2P_{2,1} = \frac{1}{7}P_{2,1} + \frac{3}{2}P_{4,1}, \qquad(11)$$

$$\frac{dP_2}{d\theta}\,\frac{dP_{2,1}}{d\theta} = \frac{3}{7}P_{2,1} - 6P_{4,1}, \tag{12}$$

$$\frac{P_{2,1}}{\sin\theta}\cot\theta\,\frac{dP_2}{d\theta} = \frac{1}{\sin\theta}\left(-\frac{9}{7}P_{2,1} - 3P_{4,1}\right), \tag{13}$$

$$P_{2,1}\frac{dP_2}{d\theta} = \frac{d}{d\theta}\left(-\frac{3}{56}P_{2,1} + \frac{3}{4}P_{4,1}\right) + \frac{1}{\sin\theta}\left(-\frac{3}{8}P_{1,1}\right), \tag{14}$$

$$P_2\frac{dP_{2,1}}{d\theta} = \frac{d}{d\theta}\left(\frac{11}{56}P_{2,1} + \frac{3}{4}P_{4,1}\right) + \frac{1}{\sin\theta}\left(\frac{3}{8}P_{1,1}\right), \tag{15}$$

$$\frac{dP_{2,1}}{d\theta}\,\frac{d^2P_2}{d\theta^2} = \frac{d}{d\theta}\left(\frac{3}{14}P_{2,1} - 3P_{4,1}\right) + \frac{1}{\sin\theta}\left(-\frac{3}{2}P_{1,1}\right), \tag{16}$$

$$\frac{dP_2}{d\theta}\,\frac{d^2P_{2,1}}{d\theta^2} = \frac{d}{d\theta}\left(\frac{3}{14}P_{2,1} - 3P_{4,1}\right) + \frac{1}{\sin\theta}\left(\frac{3}{2}P_{1,1}\right), \tag{17}$$

$$\frac{dP_2}{d\theta}\,\frac{d}{d\theta}(\operatorname{cosec}\theta\,P_{2,1}) = \frac{1}{\sin\theta}\left(\frac{12}{7}P_{2,1} - 3P_{4,1}\right), \tag{18}$$

$$P_2P_{2,2} = -\frac{2}{7}P_{2,2} + \frac{3}{2}P_{4,2}, \tag{19}$$

$$\begin{aligned}\frac{dP_2}{d\theta}\,\frac{dP_{2,2}}{d\theta} &= 2\sin\theta\,\frac{dP_2}{d\theta}\,\frac{d}{d\theta}(\operatorname{cosec}\theta\,P_{2,2})\\ &= 2P_{2,2}\cot\theta\,\frac{dP_2}{d\theta} = -\frac{6}{7}P_{2,2} - 6P_{4,2},\end{aligned} \tag{20}$$

$$P_{2,2}\frac{dP_2}{d\theta} = \frac{d}{d\theta}\left(-\frac{9}{14}P_{2,2} + \frac{3}{4}P_{4,2}\right), \tag{21}$$

$$P_2\frac{dP_{2,2}}{d\theta} = \frac{d}{d\theta}\left(\frac{5}{14}P_{2,2} + \frac{3}{4}P_{4,2}\right), \tag{22}$$

$$\frac{dP_{2,2}}{d\theta}\,\frac{d^2P_2}{d\theta^2} = \frac{dP_2}{d\theta}\,\frac{d^2P_{2,2}}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{3}{7}P_{2,2} - 3P_{4,2}\right), \tag{23}$$

$$\begin{aligned}X_{2,1}X'_{2,1} &= \left(\frac{1}{15} + \frac{1}{21}P_2 - \frac{4}{35}P_4\right)\cos(\alpha_1 - \beta_1)\\ &\quad + \left(\frac{1}{14}P_{2,2} + \frac{1}{2}P_{4,2}\right)\cos(2\phi - \alpha_1 - \beta_1),\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{dX_{2,1}}{d\theta}\,\frac{dX'_{2,1}}{d\theta} + \frac{1}{\sin^2\theta}\,\frac{dX_{2,1}}{d\phi}\,\frac{dX'_{2,1}}{d\phi} &= \left(\frac{2}{5} + \frac{1}{7}P_2 + \frac{16}{35}P_4\right)\cos(\alpha_1 - \beta_1)\\ &\quad + \left(\frac{3}{14}P_{2,2} - 2P_{4,2}\right)\cos(2\phi - \alpha_1 - \beta_1),\end{aligned} \tag{25}$$

$$\begin{aligned}X'_{2,1}\frac{dX_{2,1}}{d\theta} = X_{2,1}\frac{dX'_{2,1}}{d\theta} &= \frac{d}{d\theta}\Big[\left(\frac{1}{42}P_2 - \frac{2}{35}P_4\right)\cos(\alpha_1 - \beta_1)\\ &\quad + \left(\frac{1}{28}P_{2,2} + \frac{1}{4}P_{4,2}\right)\cos(2\phi - \alpha_1 - \beta_1)\Big],\end{aligned} \tag{26}$$

$$\frac{dX_{2,1}}{d\theta}\frac{d^2X'_{2,1}}{d\theta^2}+\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X'_{2,1})$$

$$=\frac{d}{d\theta}\Big[\Big(\frac{1}{14}P_2+\frac{8}{35}P_4\Big)\cos(\alpha_1-\beta_1)$$

$$+\Big(\frac{3}{28}P_{2,2}-P_{4,2}\Big)\cos(2\phi-\alpha_1-\beta_1)\Big], \qquad (27)$$

$$X'_{2,1}\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}=\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[\Big(\frac{1}{28}P_{2,2}+\frac{1}{4}P_{4,2}\Big)\cos(2\phi-\alpha_1-\beta_1)\Big]$$

$$+\frac{d}{d\theta}\Big[-\Big(\frac{1}{10}P_1+\frac{1}{15}P_3\Big)\sin(\alpha_1-\beta_1)\Big], \qquad (28)$$

$$\frac{dX_{2,1}}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X'_{2,1})+\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\Big(\frac{1}{\sin^2\theta}\frac{d^2X'_{2,1}}{d\phi^2}+\cot\theta\frac{dX'_{2,1}}{d\theta}\Big)$$

$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[\Big(\frac{3}{28}P_{2,2}-P_{4,2}\Big)\cos(2\phi-\alpha_1-\beta_1)\Big]$$

$$+\frac{d}{d\theta}\Big[\frac{1}{2}P_1\sin(\alpha_1-\beta_1)\Big], \qquad (29)$$

$$X_{2,1}X_{2,2}=\Big(\frac{2}{7}P_{2,1}-\frac{1}{2}P_{4,1}\Big)\cos(\phi+\alpha_1-\alpha_2)+\frac{1}{2}P_{4,3}\cos(3\phi-\alpha_1-\alpha_2), \qquad (30)$$

$$\frac{dX_{2,1}}{d\theta}\frac{dX_{2,2}}{d\theta}+\frac{1}{\sin^2\theta}\frac{dX_{2,1}}{d\phi}\frac{dX_{2,2}}{d\phi}$$

$$=\Big(\frac{6}{7}P_{2,1}+2P_{4,1}\Big)\cos(\phi+\alpha_1-\alpha_2)-2P_{4,3}\cos(3\phi-\alpha_1-\alpha_2), \qquad (31)$$

$$X_{2,2}\frac{dX_{2,1}}{d\theta}=\frac{d}{d\theta}\Big[\Big(\frac{15}{56}P_{2,1}-\frac{1}{4}P_{4,1}\Big)\cos(\phi+\alpha_1-\alpha_2)$$

$$+\frac{1}{4}P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\Big]+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[-\frac{1}{8}P_{1,1}\sin(\phi+\alpha_1-\alpha_2)$$

$$-\frac{1}{12}P_{3,3}\sin(3\phi-\alpha_1-\alpha_2)\Big], \qquad (32)$$

$$X_{2,1}\frac{dX_{2,2}}{d\theta}=\frac{d}{d\theta}\Big[\Big(\frac{1}{56}P_{2,1}-\frac{1}{4}P_{4,1}\Big)\cos(\phi+\alpha_1-\alpha_2)$$

$$+\frac{1}{4}P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\Big]+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[\frac{1}{8}P_{1,1}\sin(\phi+\alpha_1-\alpha_2)$$

$$+\frac{1}{12}P_{3,3}\sin(3\phi-\alpha_1-\alpha_2)\Big], \qquad (33)$$

$$\frac{dX_{2,1}}{d\theta}\frac{d^2X_{2,2}}{d\theta^2}+\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})$$

$$=\frac{d}{d\theta}\Big[\Big(\frac{3}{7}P_{2,1}+P_{4,1}\Big)\cos(\phi+\alpha_1-\alpha_2)-P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\Big]$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[P_{1,1}\sin(\phi+\alpha_1-\alpha_2)\Big], \qquad (34)$$

$$\frac{dX_{2,2}}{d\theta}\frac{d^2X_{2,1}}{d\theta^2}+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d^2}{d\theta d\phi}\left(\operatorname{cosec}\theta X_{2,1}\right)$$

$$=\frac{d}{d\theta}\left[\left(\frac{3}{7}P_{2,1}+P_{4,1}\right)\cos(\phi+\alpha_1-\alpha_2)-P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\right]$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-P_{11}\sin(\phi+\alpha_1-\alpha_2)\right], \qquad (35)$$

$$X_{2,2}\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{2}{7}P_{2,1}+\frac{1}{2}P_{4,1}\right)\cos(\phi+\alpha_1-\alpha_2)\right.$$

$$\left.+\frac{1}{6}P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\right], \qquad (36)$$

$$X_{2,1}\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(\frac{4}{7}P_{2,1}-P_{4,1}\right)\cos(\phi+\alpha_1-\alpha_2)\right.$$

$$\left.+\frac{1}{3}P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\right], \qquad (37)$$

$$\frac{dX_{2,1}}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,2})+\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d^2X_{2,2}}{d\phi^2}+\cot\theta\frac{dX_{2,2}}{d\theta}\right)$$

$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(\frac{10}{7}P_{2,1}+P_{4,1}\right)\cos(\phi+\alpha_1-\alpha_2)\right.$$

$$\left.-P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\right], \qquad (38)$$

$$\frac{dX_{2,2}}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X_{2,1})+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d^2X_{2,1}}{d\phi^2}+\cot\theta\frac{dX_{2,1}}{d\theta}\right)$$

$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{4}{7}P_{2,1}+P_{4,1}\right)\cos(\phi+\alpha_1-\alpha_2)\right.$$

$$\left.-P_{4,3}\cos(3\phi-\alpha_1-\alpha_2)\right], \qquad (39)$$

$$X_{2,2}X'_{2,2}=\left(\frac{4}{15}-\frac{8}{21}P_2+\frac{4}{35}P_4\right)\cos(\alpha_2-\beta_2)+\frac{1}{2}P_{4,4}\cos(4\phi-\alpha_2-\beta_2), \qquad (40)$$

$$\frac{dX_{2,2}}{d\theta}\frac{dX'_{2,2}}{d\theta}+\frac{1}{\sin^2\theta}\frac{dX_{2,2}}{d\phi}\frac{dX'_{2,2}}{d\phi}=\left(\frac{56}{35}-\frac{8}{7}P_2-\frac{16}{35}P_4\right)\cos(\alpha_2-\beta_2)$$

$$-2P_{4,4}\cos(4\phi-\alpha_2-\beta_2), \qquad (41)$$

$$X_{2,2}\frac{dX'_{2,2}}{d\theta}=X'_{2,2}\frac{dX_{2,2}}{d\theta}=\frac{d}{d\theta}\left[\left(-\frac{4}{21}P_2+\frac{2}{35}P_4\right)\cos(\alpha_2-\beta_2)\right.$$

$$\left.+\frac{1}{4}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right], \qquad (42)$$

$$\frac{dX_{2,2}}{d\theta}\frac{d^2X'_{2,2}}{d\theta^2}+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X'_{2,2})$$

$$=\frac{d}{d\theta}\left[-\left(\frac{4}{7}P_2+\frac{8}{35}P_4\right)\cos(\alpha_2-\beta_2)-P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right], \qquad (43)$$

$$X_{2,2}\frac{1}{\sin\theta}\frac{dX'_{2.2}}{d\phi}=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{1}{4}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right]$$
$$+\frac{d}{d\theta}\left[\left(\frac{4}{5}P_1-\frac{2}{15}P_3\right)\sin(\alpha_2-\beta_2)\right], \tag{44}$$

$$\frac{dX_{2,2}}{d\theta}\frac{d^2}{d\theta d\phi}(\operatorname{cosec}\theta X'_{2,2})+\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{d^2X'_{2,2}}{d\phi^2}+\cot\theta\frac{dX'_{2,2}}{d\theta}\right)$$
$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right]$$
$$+\frac{d}{d\theta}\left[4P_1\sin(\alpha_2-\beta_2)\right]. \tag{45}$$

As a matter of fact, we do not have products such as $X_{2,1}X_{2,2}$, but two sets, viz. $X_{2,1}X'_{2,2}$ and $X'_{2,1}X_{2,2}$; the results, however, for the two sets actually occurring may be deduced from the one set actually given by writing β_2 for α_2 in the one case and β_1 for α_1 in the other.

§30. The following set of results are required in dealing with the surface forces with Φ coefficients, i. e. those coming from a stream function:

$$\frac{dP_2}{d\theta}\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}=\left(\frac{3}{5}P_{1,1}+3P_{3,1}\right)\sin(\phi-\beta_1), \tag{46}$$

$$\frac{dP_2}{d\theta}\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}=6P_{3,2}\sin(2\phi-\beta_2), \tag{47}$$

$$\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{dX'_{2,1}}{d\theta}-\frac{dX_{2,1}}{d\theta}\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}=\left(\frac{1}{5}P_1+\frac{4}{5}P_3\right)\sin(\alpha_1-\beta_1), \tag{48}$$

$$\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{dX'_{2,2}}{d\theta}-\frac{dX_{2,1}}{d\theta}\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}$$
$$=\left(-\frac{2}{5}P_{1,1}+3P_{3,1}\right)\sin(\phi+\alpha_1-\beta_2)-P_{3,3}\sin(3\phi-\alpha_1-\beta_2), \tag{49}$$

$$\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{dX'_{2,2}}{d\theta}-\frac{dX_{2,2}}{d\theta}\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}=\frac{8}{5}(P_1-P_3)\sin(\alpha_2-\beta_2), \tag{50}$$

$$P_2\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}-2\frac{dP_2}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}\right)$$
$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{23}{7}P_{2,1}+\frac{15}{2}P_{4,1}\right)\cos(\phi-\beta_1)\right], \tag{51}$$

$$P_2\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}-2\frac{dP_2}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}\right)$$
$$=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(\frac{4}{7}P_{2,2}+\frac{15}{2}P_{4,2}\right)\cos(2\phi-\beta_2)\right], \tag{52}$$

$$\frac{1}{\sin\theta}\frac{dX_{2,1}}{d\phi}\left(6P_2+2\frac{d^2P_2}{d\theta^2}\right)=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(\frac{12}{7}P_{2,1}-3P_{4,1}\right)\cos(\phi-\alpha_1)\right], \tag{53}$$

$$\frac{1}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(6P_2+2\frac{d^2P_2}{d\theta^2}\right)=\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(\frac{18}{7}P_{2,2}-3P_{4,2}\right)\cos(2\phi-\alpha_2)\right], \tag{54}$$

$$\begin{aligned} X_{2,1}\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}-2\frac{dX_{2,1}}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}\right)+\frac{2}{\sin\theta}\frac{dX_{2,1}}{d\phi}\left(3X'_{2,1}+\frac{d^2X'_{2,1}}{d\theta^2}\right)& \\ =\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{11}{28}P_{2,2}+\frac{3}{4}P_{4,2}\right)\cos(2\phi-\alpha_1-\beta_1)\right]& \\ +\frac{d}{d\theta}\left[\left(-\frac{3}{10}P_1+\frac{7}{15}P_3\right)\sin(\alpha_1-\beta_1)\right],& \end{aligned} \tag{55}$$

$$\begin{aligned} X_{2,1}\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}-2\frac{dX_{2,1}}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}\right)+\frac{2}{\sin\theta}\frac{dX_{2,1}}{d\phi}\left(3X'_{2,2}+\frac{d^2X'_{2,2}}{d\theta^2}\right)& \\ =\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\left(\frac{4}{7}P_{2,1}+6P_{4,1}\right)\cos(\phi-\beta_2+\alpha_1)\right.& \\ \left.+\frac{4}{3}P_{4,3}\cos(3\phi-\alpha_1-\beta_2)\right],& \end{aligned} \tag{56}$$

$$\begin{aligned} X_{2,2}\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}-2\frac{dX_{2,2}}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}\right)+\frac{2}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(3X'_{2,1}+\frac{d^2X'_{2,1}}{d\theta^2}\right)& \\ =\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{18}{7}P_{2,1}+\frac{9}{2}P_{4,1}\right)\cos(\phi+\beta_1-\alpha_2)\right.& \\ \left.+\frac{1}{4}P_{4,3}\cos(3\phi-\alpha_2-\beta_1)\right],& \end{aligned} \tag{57}$$

$$\begin{aligned} X_{2,2}\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}-2\frac{dX_{2,2}}{d\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}\right)+\frac{2}{\sin\theta}\frac{dX_{2,2}}{d\phi}\left(3X'_{2,2}+\frac{d^2X'_{2,2}}{d\theta^2}\right)& \\ =\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{3}{4}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right]& \\ -\frac{d}{d\theta}\left[\left(\frac{12}{5}P_1+\frac{14}{15}P_3\right)\sin(\alpha_2-\beta_2)\right],& \end{aligned} \tag{58}$$

$$\frac{dP_2}{d\theta}\left(5P_2+2\frac{d^2P_2}{d\theta^2}\right)=\frac{d}{d\theta}\left(\frac{11}{7}P_2-\frac{27}{35}P_4\right), \tag{59}$$

$$\begin{aligned} -P_2\frac{dX'_{2,1}}{d\theta}+2\frac{dP_2}{d\theta}\left(3X'_{2,1}+\frac{d^2X'_{2,1}}{d\theta^2}\right)& \\ =\frac{d}{d\theta}\left[-\left(\frac{5}{56}P_{2,1}+\frac{9}{4}P_{4,1}\right)\cos(\phi-\beta_1)\right]& \\ +\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{3}{8}P_{1,1}\sin(\phi-\beta_1)\right],& \end{aligned} \tag{60}$$

$$\begin{aligned} -P_2\frac{dX'_{2,2}}{d\theta}+2\frac{dP_2}{d\theta}\left(3X'_{2,2}+\frac{d^2X'_{2,2}}{d\theta^2}\right)& \\ =\frac{d}{d\theta}\left[-\left(\frac{71}{14}P_{2,2}+\frac{9}{4}P_{4,2}\right)\cos(2\phi-\beta_2)\right],& \end{aligned} \tag{61}$$

$$
\begin{aligned}
-X_{2,1}\frac{dP_2}{d\theta}+2\frac{dX_{2,1}}{d\theta}\left(3P_2+\frac{d^2P_2}{d\theta^2}\right)&=\frac{d}{d\theta}\left[\left(\frac{93}{56}P_{2,1}-\frac{9}{4}P_{4,1}\right)\cos(\phi-\alpha_1)\right]\\
&\quad+\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{3}{8}P_{1,1}\sin(\phi-\alpha_1)\right],
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
-X_{2,2}\frac{dP_2}{d\theta}&+2\frac{dX_{2,2}}{d\theta}\left(3P_2+\frac{d^2P_2}{d\theta^2}\right)\\
&=\frac{d}{d\theta}\left[\left(\frac{27}{14}P_{2,2}-\frac{9}{4}P_{4,2}\right)\cos(2\phi-\alpha_2)\right],
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
-X_{2,1}\frac{dX'_{2,1}}{d\theta}&+2\frac{dX_{2,1}}{d\theta}\left(3X'_{2,1}+\frac{d^2X'_{2,1}}{d\theta^2}\right)+\frac{2}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}\right)\\
&=\frac{d}{d\theta}\left[\left(\frac{11}{42}P_2+\frac{6}{35}P_4\right)\cos(\alpha_1-\beta_1)\right.\\
&\quad\left.+\left(\frac{11}{28}P_{2,2}-\frac{3}{4}P_{4,2}\right)\cos(2\phi-\alpha_1-\beta_1)\right],
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
-X_{2,1}&\frac{dX'_{2,2}}{d\theta}+2\frac{dX_{2,1}}{d\theta}\left(3X'_{2,2}+\frac{d^2X'_{2,2}}{d\theta^2}\right)+\frac{2}{\sin\theta}\frac{dX_{2,1}}{d\phi}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}\right)\\
&=\frac{d}{d\theta}\left[\left(\frac{137}{56}P_{2,1}+\frac{3}{4}P_{4,1}\right)\cos(\phi-\beta_2+\alpha_1)\right.\\
&\quad\left.-\frac{3}{4}P_{4,3}\cos(3\phi-\alpha_1-\beta_2)\right]+\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\frac{9}{8}P_{1,1}\sin(\phi-\beta_2+\alpha_1)\right.\\
&\quad\left.-\frac{7}{12}P_{3,3}\sin(3\phi-\alpha_1-\beta_2)\right],
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
-X_{2,2}&\frac{dX'_{2,1}}{d\theta}+2\frac{dX_{2,2}}{d\theta}\left(3X'_{2,1}+\frac{d^2X'_{2,1}}{d\theta^2}\right)+\frac{2}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,1}}{d\phi}\right)\\
&=\frac{d}{d\theta}\left[\left(\frac{39}{56}P_{2,1}+\frac{3}{4}P_{4,1}\right)\cos(\phi-\alpha_2+\beta_1)\right.\\
&\quad\left.-\frac{3}{4}P_{4,3}\cos(3\phi-\alpha_2-\beta_1)\right]+\frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{9}{8}P_{1,1}\sin(\phi-\alpha_2+\beta_1)\right.\\
&\quad\left.+\frac{7}{12}P_{3,3}\sin(3\phi-\alpha_2-\beta_1)\right],
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
-X_{2,2}&\frac{dX'_{2,2}}{d\theta}+2\frac{dX_{2,2}}{d\theta}\left(3X'_{2,2}+\frac{d^2X'_{2,2}}{d\theta^2}\right)+\frac{2}{\sin\theta}\frac{dX_{2,2}}{d\phi}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dX'_{2,2}}{d\phi}\right)\\
&=\frac{d}{d\theta}\left[-\left(\frac{44}{21}P_2+\frac{6}{35}P_4\right)\cos(\alpha_2-\beta_2)\right.\\
&\quad\left.-\frac{3}{4}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\right].
\end{aligned}
\tag{67}
$$

In one or two cases results are wanted such as that answering to (44) with $X_{2,2}$ and $X'_{2,2}$ interposed, or that answering to (28) with $X_{2,1}$ and $X'_{2,1}$ interposed.

§31. Yet another set of results are wanted, constituting what may be called *alternative forms*, viz.

$$\frac{d}{d\theta}(P_{3,1}\cos\phi) = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{32}{35}P_{2,1}+3P_{4,1}\right)\sin\phi\right], \tag{68}$$

$$\frac{d}{d\theta}(P_{3,2}\cos 2\phi) = \frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(-\frac{2}{7}P_{2,2}+\frac{3}{2}P_{4,2}\right)\sin 2\phi\right], \tag{69}$$

$$\frac{d}{d\theta}(P_{3,3}\cos 3\phi) = \frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\sin 3\phi), \tag{70}$$

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\left[\left(P_{3,1}-\frac{3}{10}P_{1,1}\right)\sin\phi\right] = \frac{1}{2}\frac{d}{d\theta}(P_{2,1}\cos\phi), \tag{71}$$

$$\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,2}\sin 2\phi) = \frac{d}{d\theta}(P_{2,2}\cos 2\phi), \tag{72}$$

along with the corresponding results when $\sin\phi$ is written for $\cos\phi$ in the expressions differentiated with respect to θ and $(-\cos\phi)$ for $\sin\phi$ in the expressions differentiated with respect to ϕ.

The occasion for these alternative forms arises in the following way: The forces derived from the stream function are no longer completely separable from the others. In the second surface condition, for instance, we might have the expression on the left of (70) occurring with a Y or Z coefficient and at the same time the expression on the right of (70) with a W coefficient. If the identity of the two variable coefficients were overlooked, one would erroneously conclude that there were too many equations to determine the arbitrary constants in a consistent way.

§32. The absence of a separation of the two sets of surface forces shows itself in such a result as (28), which is used in the third surface condition. On the right-hand side are two sets of terms. The first set, in which the differentiation is with respect to ϕ, present themselves in the determination of the Y and Z arbitrary constants of the general solution; the second set, in which the differentiation is with respect to θ, aid in the determination of the W constants. Since $X'_{2,1}$ has a Θ coefficient, the principal terms connected with it are of the same type as those with Y and Z coefficients.

In (14), and the three following results, the ϕ factors are omitted for brevity; in actual use the forms presenting themselves are such as

$$X'_{2,1}\frac{dP_2}{d\theta} = \frac{d}{d\theta}\left[\left(-\frac{3}{56}P_{2,1}+\frac{3}{4}P_{4,1}\right)\cos(\phi-\beta_1)\right] + \frac{1}{\sin\theta}\frac{d}{d\phi}\left[-\frac{3}{8}P_{1,1}\sin(\phi-\beta_1)\right].$$

This result occurs in the second surface condition. In it the first set of terms on the right, in which the differentiation is with respect to θ, enter into the determination of the Y or Z constants. On the other hand, (55) occurs in the second surface condition with a Φ coefficient, and it is the constants of type W that have to do with the first set of terms in which the differentiation is with respect to ϕ. The principal terms are here exclusively of the W type, but there are subsidiary terms of the Y and Z type.

In all cases in these equations when there are differentiations with respect to both ϕ and θ, the set of terms put first occur in the determination of that class of arbitrary constants which would alone arise in a perfect sphere under surface forces of the specified kind. In the second set of terms in such a case there always appears P_1 or $P_{1,1}$; it may be alone or along with some other terms. Terms containing P_1 or $P_{1,1}$, as will appear subsequently, are connected with rigid body displacements and with the statical resultants of the applied forces.

§33. The occurrence of the several results is as follows: (7), (8), (11), (12), (19), (20), (24), (25), (30), (31), (40), (41) and (46)–(50) occur in the first surface condition; (9), (10), (14)–(17), (21)–(23), (26), (27), (32)–(35), (42), (43) and (51)–(58) occur in the second surface condition; (13), (18), (20), (28), (29), (36)–(39), (44), (45) and (59)–(67) occur in the third surface condition. The alternative forms (68)–(72) are required in the second and third surface conditions.

34. Referred to the fundamental directions, r, θ, ϕ, at a point on the surface (3) the direction cosines of the normal, squares and products of ε, ε_1, ε_2 being neglected, are

$$1\,,\quad -\frac{d}{d\theta}(\varepsilon P_2+\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\ \text{ and }\ -\frac{1}{\sin\theta}\,\frac{d}{d\phi}(\varepsilon P_2+\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\,.$$

Thus the surface equations take the form

$$\widehat{rr}-\frac{d}{d\theta}(\varepsilon P_2+\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\,\widehat{r\theta}-\frac{1}{\sin\theta}\,\frac{d}{d\phi}(\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\,\widehat{r\phi}$$
$$=R_2P_2+R_{2,\,1}X'_{2,\,1}+R_{2,\,2}X'_{2,\,2}\,,\qquad(73)$$

$$\widehat{r\theta}-\frac{d}{d\theta}(\varepsilon P_2+\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\,\widehat{\theta\theta}-\frac{1}{\sin\theta}\,\frac{d}{d\phi}(\varepsilon_1 X_{2,\,1}+\varepsilon_2 X_{2,\,2})\,\widehat{\theta\phi}$$
$$=\frac{d}{d\theta}(\Theta_2P_2+\Theta_{2,\,1}X'_{2,\,1}+\Theta_{2,\,2}X'_{2,\,2})+\frac{1}{\sin\theta}\,\frac{d}{d\phi}(\Phi_{2,\,1}X'_{2,\,1}+\Phi_{2,\,2}X'_{2.\,2})\,,\qquad(74)$$

$$\widehat{r\phi} - \frac{d}{d\theta}(\epsilon P_2 + \epsilon_1 X_{2,1} + \epsilon_2 X \quad)\widehat{\theta\phi} - \frac{1}{\sin\theta}\frac{d}{d\phi}(\epsilon_1 X_{2,1} + \epsilon_2 X_{2,2})\widehat{\phi\phi}$$
$$= \frac{1}{\sin\theta}\frac{d}{d\phi}(\Theta_{2,1}X'_{2,1} + \Theta_{2,2}X'_{2,2}) - \frac{d}{d\theta}(\Phi_2 P_2 + \Phi_{2,1}X'_{2,1} + \Phi_{2,2}X'_{2,2}). \quad (75)$$

The notation of (6)–(17) and of (24), Section I, is also employed in the present problem. Thus for the values, when $r = a$, of the stresses arising from the principal terms, we have

$$\widehat{rr} = \overline{L}P_2 + \overline{L}_1 X'_{2,1} + \overline{L}_2 X'_{2,2}, \quad (76)$$

$$\widehat{\theta\theta} = \overline{N}P_2 + \overline{N}_1 X'_{2,1} + \overline{N}_2 X'_{2,2} + \frac{d^2}{d\theta^2}(\overline{O}P_2 + \overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2})$$
$$+ 2na\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\phi}\right)(W_2 P_2 + W_{2,1}X'_{2,1} + W_{2,2}X'_{2,2}), \quad (77)$$

$$\widehat{\phi\phi} = \overline{N}P_2 + \overline{N}_1 X'_{2,1} + \overline{N}_2 X'_{2,2} + \left(\frac{1}{\sin^2\theta}\frac{d^2}{d\phi^2} + \cot\theta\frac{d}{d\theta}\right)(\overline{O}P_2 + \overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2})$$
$$- 2na\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\phi}\right)(W_2 P_2 + W_{2,1}X'_{2,1} + W_{2,2}X'_{2,2}), \quad (78)$$

$$\widehat{r\theta} = \frac{d}{d\theta}(\overline{M}P_2 + \overline{M}_1 X'_{2,1} + \overline{M}_2 X'_{2,2}) + na\frac{1}{\sin\theta}\frac{d}{d\phi}(W_{2,1}X'_{2,1} + W_{2,2}X'_{2,2}), \quad (79)$$

$$\widehat{r\phi} = \frac{1}{\sin\theta}\frac{d}{d\phi}(\overline{M}_1 X'_{2,1} + \overline{M}_2 X'_{2,2}) - na\frac{d}{d\theta}(W_2 P_2 + W_{2,1}X'_{2,1} + W_{2,2}X'_{2,2}), \quad (80)$$

$$\widehat{\theta\phi} = \frac{d^2}{d\theta d\phi}[\operatorname{cosec}\theta(\overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2})]$$
$$- na\left(6 + 2\frac{d^2}{d\theta^2}\right)(W_2 P_2 + W_{2,1}X'_{2,1} + W_{2,2}X'_{2,2}). \quad (81)$$

As the applied forces contain only harmonics of the second degree, I have economized one suffix in $\overline{L}$, etc. Thus, for instance, above

$$\overline{L} = L_2 V_2 + A_2 Y_2 + B_2 Z_2,$$
$$\overline{O}_2 = O_2 V_{2,2} + A_2 Y_{2,2} + B_2 Z_{2,2}.$$

If applied forces depending on harmonics of other degrees had occurred, we should have had to write $\overline{L}_2$ for $\overline{L}$ and $\overline{O}_{2,2}$ for $\overline{O}_2$ in the above.

§35. Every Y, Z and W constant must be determined by means of the surface conditions. As in the previous problems, there are principal terms containing the harmonics which occur in the applied forces, and these alone would exist if the surface were truly spherical. There are also subsidiary terms depending on a large number of harmonics of different degrees.

In the case of harmonics the same in form as those in the applied forces, the notation $Y_2 + \delta Y_2$, etc., is used for the arbitrary constants. Here Y_2 represents the first approximation when terms in ε, ε_1, ε_2 are neglected. For the constant coefficients of harmonics which would not exist if the surface were truly spherical, one letter only is used; e. g. Y_4 appears alone as the coefficient of P_4.

In the case of harmonics containing ϕ, undashed letters are used in the subsidiary terms containing a cosine, and dashed letters in those containing a sine, e. g. $Y_{4,2}P_{4,2}\cos 2\phi$, $Y'_{4,2}P_{4,2}\sin 2\phi$.

The subsidiary harmonics required are learned by inspection of the harmonics occurring in the left of the surface equations (73), (74), (75) when $a(1+\varepsilon P_2+\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})$ is written for r and account is taken only of the principal terms in the stresses.

§36. In the following identity (82) the left-hand side gives the original form of the terms which contain ε, ε_1 or ε_2 explicitly on the left-hand side of (73), while the right-hand side gives the final form when use is made of such of the results (7)–(67) as apply to the first surface condition:

$$
\begin{aligned}
&(\varepsilon P_2+\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})\, a\frac{d}{da}(\overline{L}P_2+\overline{L}_1 X'_{2,1}+\overline{L}_2 X'_{2,2}) \\
&\quad -\frac{d}{d\theta}(\varepsilon P_2+\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})\frac{d}{d\theta}(\overline{M}P_2+\overline{M}_1 X'_{2,1}+\overline{M}_2 X'_{2,2}) \\
&\quad -\frac{1}{\sin\theta}\frac{d}{d\phi}(\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})\frac{1}{\sin\theta}\frac{d}{d\phi}(\overline{M}_1 X'_{2,1}+\overline{M}_2 X'_{2,2}) \\
&\quad -\frac{d}{d\theta}(\varepsilon P_2+\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})\frac{na}{\sin\theta}\frac{d}{d\phi}(W_{2,1}X'_{2,1}+W_{2,2}X'_{2,2}) \\
&\quad +\frac{1}{\sin\theta}\frac{d}{d\phi}(\varepsilon_1 X_{2,1}+\varepsilon_2 X_{2,2})\, na\frac{d}{d\theta}(W_2 P_2+W_{2,1}X'_{2,1}+W_{2,2}X'_{2,2}) \\
&\quad \equiv a\frac{d\overline{L}}{da}\Big[\varepsilon\Big(\frac{1}{5}+\frac{2}{7}P_2+\frac{18}{35}P_4\Big)+\varepsilon_1\Big(\frac{1}{7}P_{2,1}+\frac{3}{2}P_{4,1}\Big)\cos(\phi-\alpha_1) \\
&\quad +\varepsilon_2\Big(-\frac{2}{7}P_{2,2}+\frac{3}{2}P_{4,2}\Big)\cos(2\phi-\alpha_2)\Big] \\
&\quad +a\frac{d\overline{L}_1}{da}\Big[\varepsilon\Big(\frac{1}{7}P_{2,1}+\frac{3}{2}P_{4,1}\Big)\cos(\phi-\beta_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon_1\Bigl\{\Bigl(\frac{1}{15}+\frac{1}{21}P_2-\frac{4}{35}P_4\Bigr)\cos(\alpha_1-\beta_1)\\
&+\Bigl(\frac{1}{14}P_{2,2}+\frac{1}{2}P_{4,2}\Bigr)\cos(2\phi-\alpha_1-\beta_1)\Bigr\}\\
&+\varepsilon_2\Bigl\{\Bigl(\frac{2}{7}P_{2,1}-\frac{1}{2}P_{4,1}\Bigr)\cos(\phi+\beta_1-\alpha_2)+\frac{1}{2}P_{4,3}\cos(3\phi-\beta_1-\alpha_2)\Bigr\}\Bigr]\\
&+a\frac{d\overline{L}_2}{da}\Bigl[\varepsilon\Bigl(-\frac{2}{7}P_{2,2}+\frac{3}{2}P_{4,2}\Bigr)\cos(2\phi-\beta_2)\\
&+\varepsilon_1\Bigl\{\Bigl(\frac{2}{7}P_{2,1}-\frac{1}{2}P_{4,1}\Bigr)\cos(\phi+\alpha_1-\beta_2)+\frac{1}{2}P_{4,3}\cos(3\phi-\alpha_1-\beta_2)\Bigr\}\\
&+\varepsilon_2\Bigl\{\Bigl(\frac{4}{15}-\frac{8}{21}P_2+\frac{4}{35}P_4\Bigr)\cos(\alpha_2-\beta_2)+\frac{1}{2}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\Bigr\}\Bigr]\\
&-\overline{M}\Bigl[\varepsilon\Bigl(\frac{6}{5}+\frac{6}{7}P_2-\frac{72}{35}P_4\Bigr)+\varepsilon_1\Bigl(\frac{3}{7}P_{2,1}-6P_{4,1}\Bigr)\cos(\phi-\alpha_1)\\
&+\varepsilon_2\Bigl(-\frac{6}{7}P_{2,2}-6P_{4,2}\Bigr)\cos(2\phi-\alpha_2)\Bigr]\\
&-\overline{M_1}\Bigl[\varepsilon\Bigl(\frac{3}{7}P_{2,1}-6P_{4,1}\Bigr)\cos(\phi-\beta_1)\\
&+\varepsilon_1\Bigl\{\Bigl(\frac{2}{5}+\frac{1}{7}P_2+\frac{16}{35}P_4\Bigr)\cos(\alpha_1-\beta_1)\\
&+\Bigl(\frac{3}{14}P_{2,2}-2P_{4,2}\Bigr)\cos(2\phi-\alpha_1-\beta_1)\Bigr\}\\
&+\varepsilon_2\Bigl\{\Bigl(\frac{6}{7}P_{2,1}+2P_{4,1}\Bigr)\cos(\phi+\beta_1-\alpha_2)-2P_{4,3}\cos(3\phi-\beta_1-\alpha_2)\Bigr\}\Bigr]\\
&-\overline{M_2}\Bigl[\varepsilon\Bigl(-\frac{6}{7}P_{2,2}-6P_{4,2}\Bigr)\cos(2\phi-\beta_2)\\
&+\varepsilon_1\Bigl\{\Bigl(\frac{6}{7}P_{2,1}+2P_{4,1}\Bigr)\cos(\phi+\alpha_1-\beta_2)-2P_{4,3}\cos(3\phi-\alpha_1-\beta_2)\Bigr\}\\
&+\varepsilon_2\Bigl\{\Bigl(\frac{8}{5}-\frac{8}{7}P_2-\frac{16}{35}P_4\Bigr)\cos(\alpha_2-\beta_2)-2P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\Bigr\}\Bigr]\\
&+naW_2\Bigl[\varepsilon_1\Bigl(\frac{3}{5}P_{1,1}+3P_{3,1}\Bigr)\sin(\phi-\alpha_1)+6\varepsilon_2P_{3,2}\sin(2\phi-\alpha_2)\Bigr]\\
&+naW_{2,1}\Bigl[-\varepsilon\Bigl(\frac{3}{5}P_{1,1}+3P_{3,1}\Bigr)\sin(\phi-\beta_1)+\varepsilon_1\Bigl(\frac{1}{5}P_1+\frac{4}{5}P_3\Bigr)\sin(\alpha_1-\beta_1)\\
&+\varepsilon_2\Bigl\{\Bigl(\frac{2}{5}P_{1,1}-3P_{3,1}\Bigr)\sin(\phi+\beta_1-\alpha_2)+P_{3,3}\sin(3\phi-\beta_1-\alpha_2)\Bigr\}\Bigr]\\
&+naW_{2,2}\Bigl[-6\varepsilon P_{3,2}\sin(2\phi-\beta_2)\\
&+\varepsilon_1\Bigl\{\Bigl(-\frac{2}{5}P_{1,1}+3P_{3,1}\Bigr)\sin(\phi+\alpha_1-\beta_2)-P_{3,3}\sin(3\phi-\alpha_1-\beta_2)\Bigr\}\\
&+\frac{8}{5}\varepsilon_2(P_1-P_3)\sin(\alpha_2-\beta_2)\Bigr]. \qquad (82)
\end{aligned}
$$

§37. When the applied forces are in equilibrium we can introduce sufficient subsidiary terms to give a perfect solution. Suppose, however, we do not at this stage introduce the conditions for equilibrium, and that in consequence there remain on the left-hand side of (73) what may be called the *uncompensated* terms $\mathrm{a}_1 \sin\theta \cos\phi + \mathrm{b}_1 \sin\theta \sin\phi + \mathrm{c}_1 \cos\theta$. Then if we add $\mathrm{a}_1 \sin\theta \cos\phi + \mathrm{b}_1 \sin\theta \sin\phi + \mathrm{c}_1 \cos\theta$ to the right-hand side of (73) and introduce the proper subsidiary terms on the left, we ought to obtain a satisfactory equation. This equation is as follows:

$$
\begin{aligned}
& A_0 Y_0 + \{A_2(Y_2+\delta Y_2) + B_2(Z_2+\delta Z_2)\} P_2 + (A_2 Y_{2,1} + B_2 Z_{2,1}) X'_{2,1} \\
& \quad + \{(A_2\delta Y_{2,1} + B_2\delta Z_{2,1})\cos\phi + (A_2\delta Y'_{2,1} + B_2\delta Z'_{2,1})\sin\phi\} P_{2,1} + (A_2 Y_{2,2} + B_2 Z_{2,2}) X'_{2,2} \\
& \quad + \{(A_2\delta Y_{2,2} + B_2\delta Z_{2,2})\cos 2\phi + (A_2\delta Y'_{2,2} + B_2\delta Z'_{2,2})\sin 2\phi\} P_{2,2} \\
& \quad + (A_3 Y_3 + B_3 Z_3) P_3 + \{(A_3 Y_{3,1} + B_3 Z_{3,1})\cos\phi + (A_3 Y'_{3,1} + B_3 Z'_{3,1})\sin\phi\} P_{3,1} \\
& \quad + \{(A_3 Y_{3,2} + B_3 Z_{3,2})\cos 2\phi + (A_3 Y'_{3,2} + B_3 Z'_{3,2})\sin 2\phi\} P_{3,2} \\
& \quad + \{(A_3 Y_{3,3} + B_3 Z_{3,3})\cos 3\phi + (A_3 Y'_{3,3} + B_3 Z'_{3,3})\sin 3\phi\} P_{3,3} \\
& \quad + (A_4 Y_4 + B_4 Z_4) P_4 + \{(A_4 Y_{4,1} + B_4 Z_{4,1})\cos\phi + (A_4 Y'_{4,1} + B_4 Z'_{4,1})\sin\phi\} P_{4,1} \\
& \quad + \{(A_4 Y_{4,2} + B_4 Z_{4,2})\cos 2\phi + (A_4 Y'_{4,2} + B_4 Z'_{4,2})\sin 2\phi\} P_{4,2} \\
& \quad + \{(A_4 Y_{4,3} + B_4 Z_{4,3})\cos 3\phi + (A_4 Y'_{4,3} + B_4 Z'_{4,3})\sin 3\phi\} P_{4,3} \\
& \quad + \{(A_4 Y_{4,4} + B_4 Z_{4,4})\cos 4\phi + (A_4 Y'_{4,4} + B_4 Z'_{4,4})\sin 4\phi\} P_{4,4} \\
& \quad + (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, a \frac{d}{da} (\bar{L} P_2 + \bar{L}_1 X'_{2,1} + \bar{L}_2 X'_{2,2}) \\
& \quad - \frac{d}{d\theta} (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \Big\{ \frac{d}{d\theta} (\bar{M} P_2 + \bar{M}_1 X'_{2,1} + \bar{M}_2 X'_{2,2}) \\
& \quad + \frac{na}{\sin\theta} \frac{d}{d\phi} (W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \Big\} \\
& \quad - \frac{1}{\sin\theta} \frac{d}{d\phi} (\varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \Big\{ \frac{1}{\sin\theta} \frac{d}{d\phi} (\bar{M}_1 X'_{2,1} + \bar{M}_2 X'_{2,2}) \\
& \quad - na \frac{d}{d\theta} (W_2 P_2 + W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \Big\} \\
& \quad = (R_2 - L_2 V_2) P_2 + (R_{2,1} - L_2 V_{2,1}) X'_{2,1} + (R_{2,2} - L_2 V_{2,2}) X'_{2,2} \\
& \quad + (\mathrm{a}_1 \cos\phi + \mathrm{b}_1 \sin\phi) P_{1,1} + C_1 P_1. \qquad (83)
\end{aligned}
$$

For the terms containing ε, ε_1 or ε_2 explicitly the equivalent value in harmonics is to be substituted from (82).

§38. In the first place we deduce from (83) as first approximations

$$A_2 Y_2 + B_2 Z_2 = R_2 - L_2 V_2 \ldots\ldots, \qquad (84)$$

$$A_2 Y_{2,1} + B_2 Z_{2,1} = R_{2,1} - L_2 V_{2,1} \ldots\ldots, \qquad (85)$$

$$A_2 Y_{2,2} + B_2 Z_{2,2} = R_{2,2} - L_2 V_{2,2} \ldots\ldots. \qquad (86)$$

In the next place we pick out the constant term and the coefficients of the several harmonics in (83) and employ the well-known result that, supposing all the terms taken to one side of the equation, the coefficients of each surface harmonic must separately vanish. In this way we get

from the constant term,

$$A_0 Y_0 = -\frac{\varepsilon}{5}\left(a\frac{d\bar{L}}{da} - 6\bar{M}\right) - \frac{\varepsilon_1}{15}\cos(\alpha_1 - \beta_1)\left(a\frac{d\bar{L}_1}{da} - 6\bar{M}_1\right) - \frac{4}{15}\varepsilon_2\cos(\alpha_2 - \beta_2)\left(a\frac{d\bar{L}_2}{da} - 6\bar{M}_2\right), \quad (87)$$

from coefficient of P_2,

$$A_2\delta Y_2 + B_2\delta Z_2 = -\frac{2}{7}\varepsilon\left(a\frac{d\bar{L}}{da} - 3\bar{M}\right) - \frac{1}{21}\varepsilon_1\cos(\alpha_1 - \beta_1)\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right) + \frac{8}{21}\varepsilon_2\cos(\alpha_2 - \beta_2)\left(a\frac{d\bar{L}_2}{da} - 3\bar{M}_2\right), \quad (88)$$

from coefficient of $P_{2,1}\cos\phi$,

$$A_2\delta Y_{2,1} + B_2\delta Z_{2,1} = -\frac{1}{7}\varepsilon\cos\beta_1\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right) - \frac{1}{7}\varepsilon_1\left\{\cos\alpha_1\left(a\frac{d\bar{L}}{da} - 3\bar{M}\right) + 2\cos(\alpha_1 - \beta_2)\left(a\frac{d\bar{L}_2}{da} - 3\bar{M}_2\right)\right\} - \frac{2}{7}\varepsilon_2\cos(\alpha_2 - \beta_1)\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right), \quad (89)$$

from coefficient of $P_{2,1}\sin\phi$,

$$A_2\delta Y'_{2,1} + B_2\delta Z'_{2,1} = -\frac{1}{7}\varepsilon\sin\beta_1\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right) - \frac{1}{7}\varepsilon_1\left\{\sin\alpha_1\left(a\frac{d\bar{L}}{da} - 3\bar{M}\right) + 2\sin(\beta_2 - \alpha_1)\left(a\frac{d\bar{L}_2}{da} - 3\bar{M}_2\right)\right\} - \frac{2}{7}\varepsilon_2\sin(\alpha_2 - \beta_1)\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right), \quad (90)$$

from coefficient of $P_{2,2}\cos 2\phi$,

$$A_2\delta Y_{2,2} + B_2\delta Z_{2,2} = \frac{2}{7}\varepsilon\cos\beta_2\left(a\frac{d\bar{L}_2}{da} - 3\bar{M}_2\right) - \frac{1}{14}\varepsilon_1\cos(\alpha_1 + \beta_1)\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right) + \frac{2}{7}\varepsilon_2\cos\alpha_2\left(a\frac{d\bar{L}}{da} - 3\bar{M}\right), \quad (91)$$

from coefficient of $P_{2,2}\sin 2\phi$,

$$A_2\delta Y'_{2,2} + B_2\delta Z'_{2,2} = \frac{2}{7}\varepsilon\sin\beta_2\left(a\frac{d\bar{L}_2}{da} - 3\bar{M}_2\right) - \frac{1}{14}\varepsilon_1\sin(\alpha_1 + \beta_1)\left(a\frac{d\bar{L}_1}{da} - 3\bar{M}_1\right) + \frac{2}{7}\varepsilon_2\sin\alpha_2\left(a\frac{d\bar{L}}{da} - 3\bar{M}\right), \quad (92)$$

from coefficient of P_3,

$$A_3Y_3+B_3Z_3=\frac{4}{5}na\{-\epsilon_1W_{2,1}\sin(\alpha_1-\beta_1)+2\epsilon_2W_{2,2}\sin(\alpha_2-\beta_2)\}, \tag{93}$$

from coefficient of $P_{3,1}\cos\phi$,

$$A_3Y_{3,1}+B_3Z_{3,1}=3na\{-\epsilon\sin\beta_1W_{2,1}+\epsilon_1(\sin\alpha_1W_2+\sin(\beta_2-\alpha_1)W_{2,2})\\ -\epsilon_2\sin(\alpha_2-\beta_1)W_{2,1}\}, \tag{94}$$

from coefficient of $P_{3,1}\sin\phi$,

$$A_3Y'_{3,1}+B_3Z'_{3,1}=3na\{\epsilon\cos\beta_1W_{2,1}-\epsilon_1(\cos\alpha_1W_2+\cos(\beta_2-\alpha_1)W_{2,2})\\ +\epsilon_2\cos(\alpha_2-\beta_1)W_{2,1}\}, \tag{95}$$

from coefficient of $P_{3,2}\cos 2\phi$,

$$A_3Y_{3,2}+B_3Z_{3,2}=6na(-\epsilon\sin\beta_2W_{2,2}+\epsilon_2\sin\alpha_2W_2), \tag{96}$$

from coefficient of $P_{3,2}\sin 2\phi$,

$$A_3Y'_{3,2}+B_3Z'_{3,2}=6na(\epsilon\cos\beta_2W_{2,2}-\epsilon_2\cos\alpha_2W_2), \tag{97}$$

from coefficient of $P_{3,3}\cos 3\phi$,

$$A_3Y_{3,3}+B_3Z_{3,3}=na\{-\epsilon_1\sin(\beta_2+\alpha_1)W_{2,2}+\epsilon_2\sin(\alpha_2+\beta_1)]W_{2,1}\}, \tag{98}$$

from coefficient of $P_{3,3}\sin 3\phi$,

$$A_3Y'_{3,3}+B_3Z'_{3,3}=na\{\epsilon_1\cos(\beta_2+\alpha_1)W_{2,2}-\epsilon_2\cos(\alpha_2+\beta_1)W_{2,1}\}, \tag{99}$$

from coefficient of P_4,

$$A_4Y_4+B_4Z_4=-\frac{18}{35}\epsilon\left(a\frac{d\overline{L}}{da}+4\overline{M}\right)+\frac{4}{35}\epsilon_1\cos(\alpha_1-\beta_1)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right)\\ -\frac{4}{35}\epsilon_2\cos(\alpha_2-\beta_2)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right), \tag{100}$$

from coefficient of $P_{4,1}\cos\phi$,

$$A_4Y_{4,1}+B_4Z_{4,1}=-\frac{3}{2}\epsilon\cos\beta_1\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right)+\epsilon_1\left\{-\frac{3}{2}\cos\alpha_1\left(a\frac{d\overline{L}}{da}+4\overline{M}\right)\right.\\ \left.+\frac{1}{2}\cos(\beta_2-\alpha_1)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)\right\}+\frac{1}{2}\epsilon_2\cos(\alpha_2-\beta_1)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right), \tag{101}$$

from coefficient of $P_{4,1}\sin\phi$,

$$A_4Y'_{4,1}+B_4Z'_{4,1}=-\frac{3}{2}\epsilon\sin\beta_1\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right)+\epsilon_1\left\{-\frac{3}{2}\sin\alpha_1\left(a\frac{d\overline{L}}{da}+4\overline{M}\right)\right.\\ \left.+\frac{1}{2}\sin(\beta_2-\alpha_1)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)\right\}+\frac{1}{2}\epsilon_2\sin(\alpha_2-\beta_1)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right), \tag{102}$$

from coefficient of $P_{4,2}\cos 2\phi$,

$$A_4Y_{4,2}+B_4Z_{4,2}=-\frac{3}{2}\varepsilon\cos\beta_2\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)$$
$$-\frac{1}{2}\varepsilon_1\cos(\alpha_1+\beta_1)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right)-\frac{3}{2}\varepsilon_2\cos\alpha_2\left(a\frac{d\overline{L}}{da}+4\overline{M}\right),\quad(103)$$

from coefficient of $P_{4,2}\sin 2\phi$,

$$A_4Y'_{4,2}+B_4Z'_{4,2}=-\frac{3}{2}\varepsilon\sin\beta_2\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)$$
$$-\frac{1}{2}\varepsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right)-\frac{3}{2}\varepsilon_2\sin\alpha_2\left(a\frac{d\overline{L}}{da}+4\overline{M}\right),\quad(104)$$

from coefficient of $P_{4,3}\cos 3\phi$,

$$A_4Y_{4,3}+B_4Z_{4,3}=-\frac{1}{2}\varepsilon_1\cos(\alpha_1+\beta_2)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)$$
$$-\frac{1}{2}\varepsilon_2\cos(\beta_1+\alpha_2)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right),\quad(105)$$

from coefficient of $P_{4,3}\sin 3\phi$,

$$A_4Y'_{4,3}+B_4Z'_{4,3}=-\frac{1}{2}\varepsilon_1\sin(\alpha_1+\beta_2)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right)$$
$$-\frac{1}{2}\varepsilon_2\sin(\beta_1+\alpha_2)\left(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\right),\quad(106)$$

from coefficient of $P_{4,4}\cos 4\phi$,

$$A_4Y_{4,4}+B_4Z_{4,4}=-\frac{1}{2}\varepsilon_2\cos(\alpha_2+\beta_2)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right),\quad(107)$$

from coefficient of $P_{4,4}\sin 4\phi$,

$$A_4Y'_{4,4}+B_4Z'_{4,4}=-\frac{1}{2}\varepsilon_2\sin(\alpha_2+\beta_2)\left(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\right).\quad(108)$$

Lastly, coming to the uncompensated terms, we have

from coefficient of $P_{1,1}\cos\phi$,

$$\mathrm{a}_1=\frac{1}{5}na\left[3(\varepsilon\sin\beta_1W_{2,1}-\varepsilon_1\sin\alpha_1W_2)\right.$$
$$\left.+2\{\varepsilon_1\sin(\beta_2-\alpha_1)W_{2,2}-\varepsilon_2\sin(\alpha_2-\beta_1)W_{2,1}\}\right],\quad(109)$$

from coefficient of $P_{1,1}\sin\phi$,

$$\mathrm{b}_1=\frac{1}{5}na\left[3(\varepsilon_1\cos\alpha_1W_2-\varepsilon\cos\beta_1W_{2,1})\right.$$
$$\left.+2\{\varepsilon_2\cos(\alpha_2-\beta_1)W_{2,1}-\varepsilon_1\cos(\beta_2-\alpha_1)W_{2,2}\}\right],\quad(110)$$

from coefficient of P_1,

$$\mathrm{c}_1=\frac{1}{5}na\{\varepsilon_1\sin(\alpha_1-\beta_1)W_{2,1}+8\varepsilon_2\sin(\alpha_2-\beta_2)W_{2,2}\}.\quad(111)$$

§39. In the second surface condition (74), that portion of the terms on the left which contains ε, ε_1 or ε_2 explicitly is in its original form the left-hand side, and in its final form—as supplied by the use of such of the results (7) to (67) as apply to the second surface condition—the right-hand side of the following identity:

$$
\begin{aligned}
&(\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, a \frac{d}{da} \frac{d}{d\theta} (\overline{M} P_2 + \overline{M}_1 X'_{2,1} + \overline{M}_2 X'_{2,2}) \\
&\quad - \frac{d}{d\theta} (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \Big\{ \overline{N} P_2 + \overline{N}_1 X'_{2,1} + \overline{N}_2 X'_{2,2} \\
&\quad + \frac{d^2}{d\theta^2} (\overline{O} P_2 + \overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2}) \Big\} \\
&\quad - \frac{1}{\sin\theta} \frac{d}{d\phi} (\varepsilon_1 X_2 + \varepsilon_2 X_{2,2}) \frac{d^2}{d\theta d\phi} \{ \operatorname{cosec}\theta\, (\overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2}) \} \\
&\quad + (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, a \frac{d}{da} \Big\{ na \frac{1}{\sin\theta} \frac{d}{d\phi} (W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \Big\} \\
&\quad - \frac{d}{d\theta} (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, 2na \frac{d}{d\theta} \Big(\frac{1}{\sin\theta} \frac{d}{d\phi} \Big) (W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \\
&\quad - \frac{1}{\sin\theta} \frac{d}{d\phi} (\varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \Big\{ - na \Big(6 + 2 \frac{d^2}{d\theta^2} \Big) (W_2 P_2 + W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \Big\} \\
&\equiv a \frac{d\overline{M}}{da} \Big[\varepsilon \frac{d}{d\theta} \Big(\frac{1}{7} P_2 + \frac{9}{35} P_4 \Big) + \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(- \frac{3}{56} P_{2,1} + \frac{3}{4} P_{4,1} \Big) \cos(\phi - \alpha_1) \\
&\quad + \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{3}{5} P_{1,1} \sin(\phi - \alpha_1) \Big) \Big\} \\
&\quad + \varepsilon_2 \frac{d}{d\theta} \Big(- \frac{9}{14} P_{2,2} + \frac{3}{4} P_{4,2} \Big) \cos(2\phi - \alpha_2) \Big] \\
&\quad + a \frac{d\overline{M}_1}{da} \Big[\varepsilon \Big\{ \frac{d}{d\theta} \Big(\frac{11}{56} P_{2,1} + \frac{3}{4} P_{4,1} \Big) \cos(\phi - \beta_1) + \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{3}{8} P_{1,1} \sin(\phi - \beta_1) \Big) \Big\} \\
&\quad + \varepsilon_1 \frac{d}{d\theta} \Big\{ \Big(\frac{1}{42} P_2 - \frac{2}{35} P_4 \Big) \cos(\alpha_1 - \beta_1) \\
&\quad + \Big(\frac{1}{28} P_{2,2} + \frac{1}{4} P_{4,2} \Big) \cos(2\phi - \alpha_1 - \beta_1) \Big\} \\
&\quad + \varepsilon_2 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{15}{56} P_{2,1} - \frac{1}{4} P_{4,1} \Big) \cos(\phi + \beta_1 - \alpha_2) + \frac{1}{4} P_{4,3} \cos(3\phi - \beta_1 - \alpha_2) \Big) \\
&\quad + \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{1}{8} P_{1,1} \sin(\phi + \beta_1 - \alpha_2) - \frac{1}{12} P_{3,3} \sin(3\phi - \beta_1 - \alpha_2) \Big) \Big\} \Big] \\
&\quad + a \frac{d\overline{M}_2}{da} \Big[\varepsilon \frac{d}{d\theta} \Big(\frac{5}{14} P_{2,2} + \frac{3}{4} P_{4,2} \Big) \cos(2\phi - \beta_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{1}{56} P_{2,1} - \frac{1}{4} P_{4,1} \Big) \cos(\phi - \beta_2 + \alpha_1) + \frac{1}{4} P_{4,3} \cos(3\phi - \alpha_1 - \beta_2) \Big) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{1}{8} P_{1,1} \sin(\phi - \beta_2 + \alpha_1) + \frac{1}{12} P_{3,3} \sin(3\phi - \alpha_1 - \beta_2) \Big) \Big\} \\
&+ \varepsilon_2 \frac{d}{d\theta} \Big\{ \Big(- \frac{4}{21} P_2 + \frac{2}{35} P_4 \Big) \cos(\alpha_2 - \beta_2) + \frac{1}{4} P_{4,4} \cos(4\phi - \alpha_2 - \beta_2) \Big\} \Big] \\
&- \overline{N} \Big[\varepsilon \frac{d}{d\theta} \Big(\frac{1}{7} P_2 + \frac{9}{35} P_4 \Big) + \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\frac{11}{56} P_{2,1} + \frac{3}{4} P_{4,1} \Big) \cos(\phi - \alpha_1) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{3}{8} P_{1,1} \sin(\phi - \alpha_1) \Big) \Big\} + \varepsilon_2 \frac{d}{d\theta} \Big(\frac{5}{14} P_{2,2} + \frac{3}{4} P_{4,2} \Big) \cos(2\phi - \alpha_2) \Big] \\
&- \overline{N_1} \Big[\varepsilon \Big\{ \frac{d}{d\theta} \Big(- \frac{3}{56} P_{2,1} + \frac{3}{4} P_{4,1} \Big) \cos(\phi - \beta_1) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{3}{8} P_{1,1} \sin(\phi - \beta_1) \Big) \Big\} \\
&+ \varepsilon_1 \frac{d}{d\theta} \Big\{ \Big(\frac{1}{42} P_2 - \frac{2}{35} P_4 \Big) \cos(\alpha_1 - \beta_1) \\
&+ \Big(\frac{1}{28} P_{2,2} + \frac{1}{4} P_{4,2} \Big) \cos(2\phi - \alpha_1 - \beta_1) \Big\} \\
&+ \varepsilon_2 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{1}{56} P_{2,1} - \frac{1}{4} P_{4,1} \Big) \cos(\phi - \alpha_2 + \beta_1) + \frac{1}{4} P_{4,3} \cos(3\phi - \alpha_2 - \beta_1) \Big) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{1}{8} P_{1,1} \sin(\phi - \alpha_2 + \beta_1) + \frac{1}{12} P_{3,3} \sin(3\phi - \alpha_2 - \beta_1) \Big) \Big\} \Big] \\
&- \overline{N_2} \Big[\varepsilon \frac{d}{d\theta} \Big(- \frac{9}{14} P_{2,2} + \frac{3}{4} P_{4,2} \Big) \cos(2\phi - \beta_2) \\
&+ \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{15}{56} P_{2\,1} - \frac{1}{4} P_{4,1} \Big) \cos(\phi - \beta_2 + \alpha_1) + \frac{1}{4} P_{4,3} \cos(3\phi - \beta_2 - \alpha_1) \Big) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{1}{8} P_{1,1} \sin(\phi - \beta_2 + \alpha_1) - \frac{1}{12} P_{3,3} \sin(3\phi - \beta_2 - \alpha_1) \Big) \Big\} \\
&+ \varepsilon_2 \frac{d}{d\theta} \Big\{ \Big(- \frac{4}{21} P_2 + \frac{2}{35} P_4 \Big) \cos(\alpha_2 - \beta_2) + \frac{1}{4} P_{4,4} \cos(4\phi - \alpha_2 - \beta_2) \Big\} \Big] \\
&- \overline{O} \Big[\varepsilon \frac{d}{d\theta} \Big(\frac{3}{7} P_2 - \frac{36}{35} P_4 \Big) + \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\frac{3}{14} P_{2,1} - 3 P_{4,1} \Big) \cos(\phi - \alpha_1) \\
&+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{3}{2} P_{1,1} \sin(\phi - \alpha_1) \Big) \Big\} + \varepsilon_2 \frac{d}{d\theta} \Big(- \frac{3}{7} P_{2,2} - 3 P_{4,2} \Big) \cos(2\phi - \alpha_2) \Big] \\
&- \overline{O_1} \Big[\varepsilon \Big\{ \frac{d}{d\theta} \Big(\frac{3}{14} P_{2,1} - 3 P_{4,1} \Big) \cos(\phi - \beta_1) + \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{3}{2} P_{1,1} \sin(\phi - \beta_1) \Big) \Big\} \\
&+ \varepsilon_1 \frac{d}{d\theta} \Big\{ \Big(\frac{1}{14} P_2 + \frac{8}{35} P_4 \Big) \cos(\alpha_1 - \beta_1) + \Big(\frac{3}{28} P_{2,2} - P_{4,2} \Big) \cos(2\phi - \alpha_1 - \beta_1) \Big\}
\end{aligned}
$$

$$+ \varepsilon_2 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{3}{7} P_{2,1} + P_{4,1} \Big) \cos(\phi - \alpha_2 + \beta_1) - P_{4,3} \cos(3\phi - \alpha_2 - \beta_1) \Big)$$

$$+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- P_{1,1} \sin(\phi - \alpha_2 + \beta_1) \Big) \Big\} \Big]$$

$$- \overline{O}_2 \Big[\varepsilon \frac{d}{d\theta} \Big(- \frac{3}{7} P_{2,2} - 3P_{4,2} \Big) \cos(2\phi - \beta_2)$$

$$+ \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\Big(\frac{3}{7} P_{2,1} + P_{4,1} \Big) \cos(\phi - \beta_2 + \alpha_1) - P_{4,3} \cos(3\phi - \beta_2 - \alpha_1) \Big)$$

$$+ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(P_{1,1} \sin(\phi - \beta_2 + \alpha_1) \Big) \Big\}$$

$$+ \varepsilon_2 \frac{d}{d\theta} \Big\{ - \Big(\frac{4}{7} P_2 + \frac{8}{35} P_4 \Big) \cos(\alpha_2 - \beta_2) - P_{4,4} \cos(4\phi - \alpha_2 - \beta_2) \Big\} \Big]$$

$$+ naW_2 \frac{1}{\sin\theta} \frac{d}{d\phi} \Big[\varepsilon_1 \Big(\frac{12}{7} P_{2,1} - 3P_{4,1} \Big) \cos(\phi - \alpha_1)$$

$$+ \varepsilon_2 \Big(\frac{18}{7} P_{2,2} - 3P_{4,2} \Big) \cos(2\phi - \alpha_2) \Big]$$

$$+ naW_{2,1} \Big[\frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \varepsilon \Big(- \frac{23}{7} P_{2,1} + \frac{15}{2} P_{4,1} \Big) \cos(\phi - \beta_1)$$

$$+ \varepsilon_1 \Big(- \frac{11}{28} P_{2,2} + \frac{3}{4} P_{4,2} \Big) \cos(2\phi - \alpha_1 - \beta_1)$$

$$+ \varepsilon_2 \Big(\Big(- \frac{18}{7} P_{2,1} + \frac{9}{2} P_{4,1} \Big) \cos(\phi - \alpha_2 + \beta_1) + \frac{1}{6} P_{4,3} \cos(3\phi - \alpha_2 - \beta_1) \Big) \Big\}$$

$$+ \frac{d}{d\theta} \Big\{ \varepsilon_1 \Big(- \frac{3}{10} P_1 + \frac{7}{15} P_3 \Big) \sin(\alpha_1 - \beta_1) \Big\} \Big]$$

$$+ naW_{2,2} \Big[\frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \varepsilon \Big(\frac{4}{7} P_{2,2} + \frac{15}{2} P_{4,2} \Big) \cos(2\phi - \beta_2)$$

$$+ \varepsilon_1 \Big(- \Big(\frac{4}{7} P_{2,1} + 6P_{4,1} \Big) \cos(\phi - \beta_2 + \alpha_1) + \frac{4}{3} P_{4,3} \cos(3\phi - \beta_2 - \alpha_1) \Big)$$

$$+ \varepsilon_2 . \frac{3}{4} P_{4,4} \cos(4\phi - \alpha_2 - \beta_2) \Big\}$$

$$+ \frac{d}{d\theta} \Big\{ - \varepsilon_2 \Big(\frac{12}{5} P_1 + \frac{14}{15} P_3 \Big) \sin(\alpha_2 - \beta_2) \Big\} \Big]. \tag{112}$$

§40. The terms remaining "uncompensated" on the left of the second surface equation may be written as

$$\cos\theta\,(a_2 \cos\phi + b_2 \sin\phi) + d_2 \cos\phi + e_2 \sin\phi + c_2 \sin\theta,$$

or

$$\frac{1}{\sin\theta} \frac{d}{d\phi} [(a_2 \sin\phi - b_2 \cos\phi) P_{2,1} + (d_2 \sin\phi - e_2 \cos\phi) P_{1,1}] - c_2 \frac{dP_1}{d\theta},$$

where a_2, b_2, etc., are constants.

In the subsidiary terms we make use of the alternative forms (68)–(72) to transform the coefficients of the arbitrary constants whose first suffix is 3, excepting $\frac{dP_3}{d\theta}$ and $\frac{1}{\sin\theta}\frac{d}{d\phi}P_{3,3}(W_{3,3}\cos 3\phi + W'_{3,3}\sin 3\phi)$. We write, for instance,

$$(C_3Y_{3,3}+D_3Z_{3,3})\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\sin 3\phi) \text{ for } (C_3Y_{3,3}+D_3Z_{3,3})\frac{d}{d\theta}(P_{3,3}\cos 3\phi),$$

and

$$W'_{3,1}\left\{\frac{1}{2}\frac{d}{d\theta}(P_{2,1}\cos\phi)+\frac{1}{\sin\theta}\frac{d}{d\phi}\left(\frac{3}{10}P_{1,1}\sin\phi\right)\right\} \text{ for } W'_{3,1}\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,1}\sin\phi).$$

Thus the full form of the second surface equation is

$$\begin{aligned}
&\{C_2(Y_2+\delta Y_2)+D_2(Z_2+\delta Z_2)\}\frac{dP_2}{d\theta}+(C_2Y_{2,1}+D_2Z_{2,1})\frac{dX'_{2,1}}{d\theta}\\
&+\{(C_2\delta Y_{2,1}+D_2\delta Z_{2,1})\cos\phi+(C_2\delta Y'_{2,1}+D_2\delta Z'_{2,1})\sin\phi\}\frac{dP_{2,1}}{d\theta}\\
&+(C_2Y_{2,2}+D_2Z_{2,2})\frac{dX'_{2,2}}{d\theta}+\{(C_2\delta Y_{2,2}+D_2\delta Z_{2,2})\cos 2\phi\\
&+(C_2\delta Y'_{2,2}+D_2\delta Z'_{2,2})\sin 2\phi\}\frac{dP_{2,2}}{d\theta}+(C_3Y_3+D_3Z_3)\frac{dP_3}{d\theta}\\
&+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big[\Big(-\frac{32}{35}P_{2,1}+3P_{4,1}\Big)\{(C_3Y_{3,1}+D_3Z_{3,1})\sin\phi\\
&-(C_3Y'_{3,1}+D_3Z'_{3,1})\cos\phi\}\\
&+\Big(-\frac{2}{7}P_{2,2}+\frac{3}{2}P_{4,2}\Big)\{(C_3Y_{3,2}+D_3Z_{3,2})\sin 2\phi-(C_3Y'_{3,2}+D_3Z'_{3,2})\cos 2\phi\}\\
&+P_{4,3}\{(C_3Y_{3,3}+D_3Z_{3,3})\sin 3\phi-(C_3Y'_{3,3}+D_3Z'_{3,3})\cos 3\phi\}\Big]\\
&+\frac{d}{d\theta}[(C_4Y_4+D_4Z_4)P_4+P_{4,1}\{(C_4Y_{4,1}+D_4Z_{4,1})\cos\phi+(C_4Y'_{4,1}+D_4Z'_{4,1})\sin\phi\}\\
&+P_{4,2}\{(C_4Y_{4,2}+D_4Z_{4,2})\cos 2\phi+(C_4Y'_{4,2}+D_4Z'_{4,2})\sin 2\phi\}\\
&+P_{4,3}\{(C_4Y_{4,3}+D_4Z_{4,3})\cos 3\phi+(C_4Y'_{4,3}+D_4Z'_{4,3})\sin 3\phi\}\\
&+P_{4,4}\{(C_4Y_{4,4}+D_4Z_{4,4})\cos 4\phi+(C_4Y'_{4,4}+D_4Z'_{4,4})\sin 4\phi\}]\\
&+na\frac{1}{\sin\theta}\frac{d}{d\phi}(W_{2,1}X'_{2,1}+W_{2,2}X'_{2,2})\\
&+na\frac{1}{\sin\theta}\frac{d}{d\phi}[P_{2,1}(\delta W_{2,1}\cos\phi+\delta W'_{2,1}\sin\phi)\\
&+P_{2,2}(\delta W_{2,2}\cos 2\phi+\delta W'_{2,2}\sin 2\phi)]\\
&+2na^2\frac{d}{d\theta}\Big[\frac{1}{2}P_{2,1}(-W_{3,1}\sin\phi+W'_{3,1}\cos\phi)
\end{aligned}$$

$$
\begin{aligned}
&+ P_{2,2}(-W_{3,2}\sin 2\phi + W'_{3,2}\cos 2\phi)\Big] \\
&+ 2na^2 \frac{1}{\sin\theta}\frac{d}{d\phi}\Big[\frac{3}{10} P_{1,1}(W_{3,1}\cos\phi + W'_{3,1}\sin\phi) \\
&+ P_{3,3}(W_{3,3}\cos 3\phi + W'_{3,3}\sin 3\phi)\Big] \\
&+ 3na^3 \frac{1}{\sin\theta}\frac{d}{d\phi}\Big[P_{4,1}(W_{4,1}\cos\phi + W'_{4,1}\sin\phi) + P_{4,2}(W_{4,2}\cos 2\phi + W'_{4,2}\sin 2\phi) \\
&+ P_{4,3}(W_{4,3}\cos 3\phi + W'_{4,3}\sin 3\phi) + P_{4,4}(W_{4,4}\cos 4\phi + W'_{4,4}\sin 4\phi)\Big] \\
&+ \text{expression on right-hand side of (112)} \\
&= \frac{d}{d\theta}\Big\{(\Theta_2 - M_2 V_2)P_2 + (\Theta_{2,1} - M_2 V_{2,1})X'_{2,1} + (\Theta_{2,2} - M_2 V_{2,2})X'_{2,2}\Big\} \\
&+ \frac{1}{\sin\theta}\frac{d}{d\phi}(\Phi_{2,1}X'_{2,1} + \Phi_{2,2}X'_{2,2}) - \mathrm{c}_2\frac{dP_1}{d\theta} \\
&+ \frac{1}{\sin\theta}\frac{d}{d\phi}\Big\{P_{2,1}(\mathrm{a}_2\sin\phi - \mathrm{b}_2\cos\phi) + P_{1,1}(\mathrm{d}_2\sin\phi - \mathrm{e}_2\cos\phi)\Big\}. \qquad (113)
\end{aligned}
$$

§41. Supposing for a moment all the terms taken to one side of the equation, we equate to zero the coefficient of each separate differential coefficient, the justification being practically the same as in the corresponding case in Sect. I.

In reality, the second and third surface conditions can hardly be treated apart. The one for instance contains terms such as

$$A_s\frac{dT_s}{d\theta}, \quad A'_s\frac{1}{\sin\theta}\frac{dT'_s}{d\phi},$$

the other corresponding terms, such as

$$A_s\frac{1}{\sin\theta}\frac{dT_s}{d\phi}, \quad A'_s\frac{dT'_s}{d\theta},$$

where for the moment A_s, A'_s represent coefficients independent of θ and ϕ answering to surface harmonics T_s, T'_s. In a good many instances, however, the identity of the corresponding coefficients in the two equations is not apparent; it thus seemed best to write down as if independent the equations obtained from the two surface conditions. Their identity will be clear enough subsequently.

§42. From the principal terms we have for the first approximations

$$C_2 Y_2 + D_2 Z_2 = \Theta_2 - M_2 V_2, \tag{114}$$

$$C_2 Y_{2,1} + D_2 Z_{2,1} = \Theta_{2,1} - M_2 V_{2,1}, \tag{115}$$

$$C_2 Y_{2,2} + D_2 Z_{2,2} = \Theta_{2,2} - M_2 V_{2,2}, \tag{116}$$

$$W_{2,1} = \frac{1}{na} \Phi_{2,1}, \tag{117}$$

$$W_{2,2} = \frac{1}{na} \Phi_{2,2}. \tag{118}$$

The same results are also, it need hardly be said, deducible as first approximations from the third surface equation. Taken along with (84), (85) and (86), they suffice to determine the first approximations Y_2, Z_2, $Y_{2,1}$, $Z_{2,1}$, $Y_{2,2}$, $Z_{2,2}$, as well as $W_{2,1}$, $W_{2,2}$. The values thus found are then to be substituted in terms containing ε, ε_1 or ε_2 to get the values of the constant coefficients of the subsidiary terms.

§43. The source of the several equations for the subsidiary constants is indicated below. We have

from coefficient of $\frac{dP_2}{d\theta}$,

$$\begin{aligned} C_2 \delta Y_2 + D_2 \delta Z_2 = &- \frac{1}{7} \varepsilon \left(a \frac{d\overline{M}}{da} - \overline{N} - 3\overline{O}\right) \\ &- \frac{1}{42} \varepsilon_1 \cos(\alpha_1 - \beta_1)\left(a \frac{d\overline{M_1}}{da} - \overline{N_1} - 3\overline{O_1}\right) \\ &+ \frac{4}{21} \varepsilon_2 \cos(\alpha_2 - \beta_2)\left(a \frac{d\overline{M_2}}{da} - \overline{N_2} - 3\overline{O_2}\right), \end{aligned} \tag{119}$$

from coefficient o $\frac{dP_{2,1}}{d\theta} \cos\phi$,

$$\begin{aligned} C_2 \delta Y_{2,1} + D_2 \delta Z_{2,1} = &\frac{1}{56} \varepsilon_1 \cos\alpha_1 \left(3a \frac{d\overline{M}}{da} + 11\overline{N} + 12\overline{O}\right) \\ &- \frac{1}{56} \varepsilon \cos\beta_1 \left(11a \frac{d\overline{M_1}}{da} + 3\overline{N_1} - 12\overline{O_1}\right) - \frac{1}{56} \varepsilon_2 \cos(\alpha_2 - \beta_1)\left(15a \frac{d\overline{M_2}}{da} - \overline{N_1} - 24\overline{O_1}\right) \\ &- \frac{1}{56} \varepsilon_1 \cos(\beta_2 - \alpha_1)\left(a \frac{d\overline{M_2}}{da} - 15\overline{N_2} - 24\overline{O_2}\right) - na^2 W'_{3,1}, \end{aligned} \tag{120}$$

from coefficient of $\dfrac{dP_{2,1}}{d\theta}\sin\phi$,

$$C_2\delta Y'_{2,1} + D_2\delta Z'_{2,1} = \frac{1}{56}\,\varepsilon_1\sin\alpha_1\left(3a\frac{d\overline{M}}{da} + 11\overline{N} + 12\overline{O}\right)$$
$$-\frac{1}{56}\,\varepsilon\sin\beta_1\left(11a\frac{d\overline{M}_1}{da} + 3\overline{N}_1 - 12\overline{O}_1\right) - \frac{1}{56}\,\varepsilon_2\sin(\alpha_2 - \beta_1)\left(15a\frac{d\overline{M}_1}{da} - \overline{N}_1 - 24\overline{O}_1\right)$$
$$-\frac{1}{56}\,\varepsilon_1\sin(\beta_2 - \alpha_1)\left(a\frac{d\overline{M}_2}{da} - 15\overline{N}_2 - 24\overline{O}_2\right) + na^2 W_{3,1}, \tag{121}$$

from coefficient of $\dfrac{dP_{2,2}}{d\theta}\cos 2\phi$,

$$C_2\delta Y_{2,2} + D_2\delta Z_{2,2} = \frac{1}{14}\,\varepsilon_2\cos\alpha_2\left(9a\frac{d\overline{M}}{da} + 5\overline{N} - 6\overline{O}\right)$$
$$-\frac{1}{28}\,\varepsilon_1\cos(\alpha_1 + \beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 - 3\overline{O}_1\right)$$
$$-\frac{1}{14}\,\varepsilon\cos\beta_2\left(5a\frac{d\overline{M}_2}{da} + 9\overline{N}_2 + 6\overline{O}_2\right) - 2na^2 W'_{3,2}, \tag{122}$$

from coefficient of $\dfrac{dP_{2,2}}{d\theta}\sin 2\phi$,

$$C_2\delta Y'_{2,2} + D_2\delta Z'_{2,2} = \frac{1}{14}\,\varepsilon_2\sin\alpha_2\left(9a\frac{d\overline{M}}{da} + 5\overline{N} - 6\overline{O}\right)$$
$$-\frac{1}{28}\,\varepsilon_1\sin(\alpha_1 + \beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 - 3\overline{O}_1\right)$$
$$-\frac{1}{14}\,\varepsilon\sin\beta_2\left(5a\frac{d\overline{M}_2}{da} + 9\overline{N}_2 + 6\overline{O}_2\right) + 2na^2 W_{3,2}, \tag{123}$$

from coefficient of $\dfrac{dP_3}{d\theta}$,

$$C_3 Y_3 + D_3 Z_3 = na\left\{-\frac{7}{15}\,\varepsilon_1\sin(\alpha_1 - \beta_1)\,W_{2,1} + \frac{14}{15}\,\varepsilon_2\sin(\alpha_2 - \beta_2)\,W_{2,2}\right\}, \tag{124}$$

from coefficient of $\dfrac{dP_4}{d\theta}$,

$$C_4 Y_4 + D_4 Z_4 = -\frac{9}{45}\,\varepsilon\left(a\frac{d\overline{M}}{da} - \overline{N} + 4\overline{O}\right) + \frac{2}{35}\,\varepsilon_1\cos(\alpha_1 - \beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 + 4\overline{O}_1\right)$$
$$-\frac{2}{35}\,\varepsilon_2\cos(\alpha_2 - \beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_2\right), \tag{125}$$

from coefficient of $\frac{dP_{4,1}}{d\theta}\cos\phi$,

$$C_4Y_{4,1}+D_4Z_{4,1}=-\frac{3}{4}\epsilon_1\cos\alpha_1\left(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\right)$$
$$+\frac{1}{4}\{-3\epsilon\cos\beta_1+\epsilon_2\cos(\alpha_2-\beta_1)\}\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)$$
$$+\frac{1}{4}\epsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \tag{126}$$

from coefficient of $\frac{dP_{4,1}}{d\theta}\sin\phi$,

$$C_4Y'_{4,1}+D_4Z'_{4,1}=-\frac{3}{4}\epsilon_1\sin\alpha_1\left(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\right)$$
$$+\frac{1}{4}\{-3\epsilon\sin\beta_1+\epsilon_2\sin(\alpha_2-\beta_1)\}\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)$$
$$+\frac{1}{4}\epsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \tag{127}$$

from coefficient of $\frac{dP_{4,2}}{d\theta}\cos 2\phi$,

$$C_4Y_{4,2}+D_4Z_{4,2}=-\frac{3}{4}\epsilon_2\cos\alpha_2\left(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\right)$$
$$-\frac{1}{4}\epsilon_1\cos(\alpha_1+\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)-\frac{3}{4}\epsilon\cos\beta_2\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \tag{128}$$

from coefficient of $\frac{dP_{4,2}}{d\theta}\sin 2\phi$,

$$C_4Y'_{4,2}+D_4Z'_{4,2}=-\frac{3}{4}\epsilon_2\sin\alpha_2\left(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\right)$$
$$-\frac{1}{4}\epsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)-\frac{3}{4}\epsilon\sin\beta_2\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \tag{129}$$

from coefficient of $\frac{dP_{4,3}}{d\theta}\cos 3\phi$,

$$C_4Y_{4,3}+D_4Z_{4,3}=-\frac{1}{4}\epsilon_2\cos(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)$$
$$-\frac{1}{4}\epsilon_1\cos(\alpha_1+\beta_2)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \tag{130}$$

from coefficient of $\frac{dP_{4,3}}{d\theta}\sin 3\phi$,

$$C_4 Y'_{4,3} + D_4 Z'_{4,3} = -\frac{1}{4}\varepsilon_2 \sin(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 + 4\overline{O}_1\right)$$
$$-\frac{1}{4}\varepsilon_1 \sin(\alpha_1+\beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_2\right), \qquad (131)$$

from coefficient of $\frac{dP_{4,4}}{d\theta}\cos 4\phi$,

$$C_4 Y_{4,4} + D_4 Z_{4,4} = -\frac{1}{4}\varepsilon_2 \cos(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_3\right), \qquad (132)$$

from coefficient of $\frac{dP_{4,4}}{d\theta}\sin 4\phi$,

$$C_4 Y'_{4,4} + D_4 Z'_{4,4} = -\frac{1}{4}\varepsilon_2 \sin(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_2\right), \qquad (133)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,1}\cos\phi)$,

$$na\delta W_{2,1} = -\frac{32}{35}(C_3 Y'_{3,1} + D_3 Z'_{3,1}) - \mathrm{b}_2 - \frac{12}{7}na\varepsilon_1 \cos\alpha_1 W_2$$
$$+ naW_{2,1}\left\{\frac{23}{7}\varepsilon\cos\beta_1 + \frac{18}{7}\varepsilon_2\cos(\alpha_2-\beta_1)\right\} + \frac{4}{7}na\varepsilon_1\cos(\beta_2-\alpha_1)W_{2,2}, \qquad (134)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2.1}\sin\phi)$,

$$na\delta W'_{2,1} = \frac{32}{35}(C_3 Y_{3,1} + D_3 Z_{3,1}) + \mathrm{a}_2 - \frac{12}{7}na\varepsilon_1 \sin\alpha_1 W_2$$
$$+ naW_{2,1}\left\{\frac{23}{7}\varepsilon\sin\beta_1 + \frac{18}{7}\varepsilon_2\sin(\alpha_2-\beta_1)\right\} + \frac{4}{7}na\varepsilon_1\sin(\beta_2-\alpha_1)W_{2,2}, \qquad (135)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,2}\cos 2\phi)$,

$$na\delta W_{2,2} = -\frac{2}{7}(C_3 Y'_{3,2} + D_3 Z'_{3,2}) - \frac{18}{7}na\varepsilon_2\cos\alpha_2 W_2$$
$$+ \frac{11}{28}na\varepsilon_1\cos(\alpha_1+\beta_1)W_{2,1} - \frac{4}{7}na\varepsilon\cos\beta_2 W_{2,2}, \qquad (136)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,2}\sin 2\phi)$,

$$na\delta W'_{2,2} = \frac{2}{7}(C_3 Y_{3,2} + D_3 Z_{3,2}) - \frac{18}{7}na\varepsilon_2\sin\alpha_2 W_2$$
$$+ \frac{11}{28}na\varepsilon_1\sin(\alpha_1+\beta_1)W_{2,1} - \frac{4}{7}na\varepsilon\sin\beta_2 W_{2,2}, \qquad (137)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,3}\cos 3\phi)$,

$$2na^2 W_{3,3} = -\frac{1}{12}\epsilon_2 \sin(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da}+\overline{N}_1\right) + \frac{1}{12}\epsilon_1\sin(\alpha_1+\beta_2)\left(a\frac{d\overline{M}_2}{da}+\overline{N}_2\right), \tag{138}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,3}\sin 3\phi)$,

$$2na^2 W'_{3,3} = \frac{1}{12}\epsilon_2 \cos(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da}+\overline{N}_1\right) - \frac{1}{12}\epsilon_1\cos(\alpha_1+\beta_2)\left(a\frac{d\overline{M}_2}{da}+\overline{N}_2\right), \tag{139}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,1}\cos\phi)$,

$$3na^3 W_{4,1} = 3(C_3Y'_{3,1}+D_3Z'_{3,1}) + 3na\epsilon_1\cos\alpha_1 W_2 - naW_{2,1}\left\{\frac{15}{2}\epsilon\cos\beta_1+\frac{9}{2}\epsilon_2\cos(\alpha_2-\beta_1)\right\} + 6na\epsilon_1\cos(\beta_2-\alpha_1)W_{2,2}, \tag{140}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,1}\sin\phi)$,

$$3na^3 W'_{4,1} = -3(C_3Y_{3,1}+D_3Z_{3,1}) + 3na\epsilon_1\sin\alpha_1 W_2 - naW_{2,1}\left\{\frac{15}{2}\epsilon\sin\beta_1+\frac{9}{2}\epsilon_2\sin(\alpha_2-\beta_1)\right\} + 6na\epsilon_1\sin(\beta_2-\alpha_1)W_{2,2}, \tag{141}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,2}\cos 2\phi)$,

$$3na^3 W_{4,2} = \frac{3}{2}(C_3Y'_{3,2}+D_3Z'_{3,2}) + 3na\epsilon_2\cos\alpha_2 W_2 - \frac{3}{4}na\epsilon_1\cos(\alpha_1+\beta_1)W_{2,1} - \frac{15}{2}na\epsilon\cos\beta_2 W_{2,2}, \tag{142}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,2}\sin 2\phi)$,

$$3na^3 W'_{4,2} = -\frac{3}{2}(C_3Y_{3,2}+D_3Z_{3,2}) + 3na\epsilon_2\sin\alpha_2 W_2 - \frac{3}{4}na\epsilon_1\sin(\alpha_1+\beta_1)W_{2,1} - \frac{15}{2}na\epsilon\sin\beta_2 W_{2,2}, \tag{143}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\cos 3\phi)$,

$$3na^3 W_{4,3} = C_3Y'_{3,3} + D_3Z'_{3,3} - \frac{1}{6}na\epsilon_2\cos(\alpha_2+\beta_1)W_{2,1} - \frac{4}{3}na\epsilon_1\cos(\alpha_1+\beta_2)W_{2,2}, \tag{144}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\sin 3\phi)$,

$$3na^3 W'_{4,3} = -C_3 Y_{3,3} - D_3 Z_{3,3} - \frac{1}{6} na\varepsilon_2 \sin(\alpha_2 + \beta_1) W_{2,1}$$
$$- \frac{4}{3} na\varepsilon_1 \sin(\alpha_1 + \beta_2) W_{2,2}, \qquad (145)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,4}\cos 4\phi)$,

$$3na^3 W_{4,4} = -\frac{3}{4} na\varepsilon_2 \cos(\alpha_2 + \beta_2) W_{2,2}, \qquad (146)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,4}\sin 4\phi)$,

$$3na^3 W'_{4,4} = -\frac{3}{4} na\varepsilon_2 \sin(\alpha_2 + \beta_2) W_{2,2}. \qquad (147)$$

Lastly, coming to the harmonics of the first degree which represent entirely uncompensated terms, we have

from coefficient of $\frac{dP_1}{d\theta}$,

$$\mathrm{c}_2 = \frac{3}{10} na\varepsilon_1 \sin(\alpha_1 - \beta_1) W_{2,1} + \frac{12}{5} na\varepsilon_2 \sin(\alpha_2 - \beta_2) W_{2,2}, \qquad (148)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{1,1}\cos\phi)$,

$$\mathrm{e}_2 = \frac{3}{8}\left\{-\varepsilon_1 \sin\alpha_1 \left(a\frac{d\overline{M}}{da} + \overline{N} - 4\overline{O}\right) + \varepsilon \sin\beta_1 \left(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 4\overline{O}_1\right)\right\}$$
$$+ \frac{1}{8}\left\{-\varepsilon_2 \sin(\alpha_2 - \beta_1)\left(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 8\overline{O}_1\right)\right.$$
$$\left. + \varepsilon_1 \sin(\beta_2 - \alpha_1)\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2 - 8\overline{O}_2\right)\right\} - \frac{3}{5} na^2 W_{3,1}, \qquad (149)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{1,1}\sin\phi)$,

$$\mathrm{d}_2 = \frac{3}{8}\left\{-\varepsilon_1 \cos\alpha_1 \left(a\frac{d\overline{M}}{da} + \overline{N} - 4\overline{O}\right) + \varepsilon \cos\beta_1 \left(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 4\overline{O}_1\right)\right\}$$
$$+ \frac{1}{8}\left\{-\varepsilon_2 \cos(\alpha_2 - \beta_1)\left(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 8\overline{O}_1\right)\right.$$
$$\left. + \varepsilon_1 \cos(\beta_2 - \alpha_1)\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2 - 8\overline{O}_2\right)\right\} + \frac{3}{5} na^2 W'_{3,1}. \qquad (150)$$

Further consideration of the second surface equation will be postponed until the third surface equation has been worked out.

§44. Referring to (75) and the expressions (78), (80) and (81) for the stresses, we see that the terms containing ε, ε_1 or ε_2 explicitly on the left of the third surface equation are

$$(\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, a \frac{d}{da}\, \frac{1}{\sin\theta}\, \frac{d}{d\phi} (\overline{M}_1 X'_{2,1} + \overline{M}_2 X'_{2,2})$$

$$- \frac{d}{d\theta} (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \frac{d^2}{d\theta d\phi} \{\operatorname{cosec}\theta\, (\overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2})\}$$

$$- \frac{1}{\sin\theta}\, \frac{d}{d\phi} (\varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2}) \Big\{ \overline{N} P_2 + \overline{N}_1 X'_{2,1} + \overline{N}_2 X'_{2,2}$$

$$+ \Bigl(\frac{1}{\sin^2\theta}\, \frac{d}{d\phi^2} + \cot\theta \frac{d}{d\theta}\Bigr)(\overline{O} P_2 + \overline{O}_1 X'_{2,1} + \overline{O}_2 X'_{2,2}) \Big\}$$

$$- (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, a \frac{d}{da} \Big\{ na \frac{d}{d\theta} (W_2 P_2 + W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2}) \Big\}$$

$$+ \frac{d}{d\theta} (\varepsilon P_2 + \varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, na \Bigl(6 + 2 \frac{d^2}{d\theta^2}\Bigr)(W_2 P_2 + W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2})$$

$$+ \frac{1}{\sin\theta}\, \frac{d}{d\phi} (\varepsilon_1 X_{2,1} + \varepsilon_2 X_{2,2})\, 2na \frac{d}{d\theta} \Bigl(\frac{1}{\sin\theta}\, \frac{d}{d\phi}\Bigr)(W_{2,1} X'_{2,1} + W_{2,2} X'_{2,2})$$

$$\equiv a \frac{d\overline{M}_1}{da} \Big[\varepsilon \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Big\{ \Bigl(\frac{1}{7} P_{2,1} + \frac{3}{2} P_{4,1}\Bigr) \cos(\phi - \beta_1) \Big\}$$

$$+ \varepsilon_1 \Big\{ \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Bigl(\Bigl(\frac{1}{28} P_{2,2} + \frac{1}{4} P_{4,2}\Bigr) \cos(2\phi - \alpha_1 - \beta_1)\Bigr)$$

$$+ \frac{d}{d\theta} \Bigl(\frac{1}{10} P_1 + \frac{1}{15} P_3\Bigr) \sin(\alpha_1 - \beta_1) \Big\}$$

$$+ \varepsilon_2 \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Big\{ \Bigl(-\frac{2}{7} P_{2,1} + \frac{1}{2} P_{4,1}\Bigr) \cos(\phi + \beta_1 - \alpha_2)$$

$$+ \frac{1}{6} P_{4,3} \cos(3\phi - \alpha_2 - \beta_1) \Big\} \Big]$$

$$+ a \frac{d\overline{M}_2}{da} \Big[\varepsilon \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Big\{ \Bigl(-\frac{2}{7} P_{2,2} + \frac{3}{2} P_{4,2}\Bigr) \cos(2\phi - \beta_2) \Big\}$$

$$+ \varepsilon_1 \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Big\{ \Bigl(\frac{4}{7} P_{2,1} - P_{4,1}\Bigr) \cos(\phi + \alpha_1 - \beta_2) + \frac{1}{3} P_{4,3} \cos(3\phi - \alpha_1 - \beta_2) \Big\}$$

$$+ \varepsilon_2 \Big\{ \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Bigl(\frac{1}{4} P_{4,4} \cos(4\phi - \alpha_2 - \beta_2)\Bigr) + \frac{d}{d\theta} \Bigl(\frac{4}{5} P_1 - \frac{2}{15} P_3\Bigr) \sin(\alpha_2 - \beta_2) \Big\} \Big]$$

$$- \overline{N} \frac{1}{\sin\theta}\, \frac{d}{d\phi} \Big[\varepsilon_1 \Bigl(\frac{1}{7} P_{2,1} + \frac{3}{2} P_{4,1}\Bigr) \cos(\phi - \alpha_1)$$

$$+ \varepsilon_2 \Big(- \frac{2}{7} P_{2,2} + \frac{3}{2} P_{4,2} \Big) \cos (2\phi - \alpha_2) \Big]$$

$$- \overline{N}_1 \Big[\varepsilon_1 \Big\{ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\Big(\frac{1}{28} P_{2,2} + \frac{1}{4} P_{4,2} \Big) \cos (2\phi - \alpha_1 - \beta_1) \Big)$$

$$- \frac{d}{d\theta} \Big(\frac{1}{10} P_1 + \frac{1}{15} P_3 \Big) \sin (\alpha_1 - \beta_1) \Big\}$$

$$+ \varepsilon_2 \frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \Big(\frac{4}{7} P_{2,1} - P_{4,1} \Big) \cos (\phi + \beta_1 - \alpha_2) + \frac{1}{3} P_{4,3} \cos (3\phi - \alpha_2 - \beta_1) \Big\} \Big]$$

$$- \overline{N}_2 \Big[\varepsilon_1 \frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \Big(- \frac{2}{7} P_{2,1} + \frac{1}{2} P_{4,1} \Big) \cos (\phi + \alpha_1 - \beta_2)$$

$$+ \frac{1}{6} P_{4,3} \cos (3\phi - \alpha_1 - \beta_2) \Big\}$$

$$+ \varepsilon_2 \Big\{ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\frac{1}{4} P_{4,4} \cos (4\phi - \alpha_2 - \beta_2) \Big)$$

$$+ \frac{d}{d\theta} \Big(- \frac{4}{5} P_1 + \frac{2}{15} P_3 \Big) \sin (\alpha_2 - \beta_2) \Big\} \Big]$$

$$- \overline{O} \frac{1}{\sin\theta} \frac{d}{d\phi} \Big[- \varepsilon_1 \Big(\frac{9}{7} P_{2,1} + 3 P_{4,1} \Big) \cos (\phi - \alpha_1)$$

$$- \varepsilon_2 \Big(\frac{3}{7} P_{2,2} + 3 P_{4,2} \Big) \cos (2\phi - \alpha_2) \Big]$$

$$- \overline{O}_1 \Big[\varepsilon \frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \Big(\frac{12}{7} P_{2,1} - 3 P_{4,1} \Big) \cos (\phi - \beta_1) \Big\}$$

$$+ \varepsilon_1 \Big\{ \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\Big(\frac{3}{28} P_{2,2} - P_{4,2} \Big) \cos (2\phi - \alpha_1 - \beta_1) \Big) + \frac{1}{2} \frac{dP_1}{d\theta} \sin (\alpha_1 - \beta_1) \Big\}$$

$$+ \varepsilon_2 \frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \Big(- \frac{4}{7} P_{2,1} + P_{4,1} \Big) \cos (\phi + \beta_1 - \alpha_2) - P_{4,3} \cos (3\phi - \beta_1 - \alpha_2) \Big\} \Big]$$

$$- \overline{O}_2 \Big[- \varepsilon \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(\Big(\frac{3}{7} P_{2,2} + 3 P_{4,2} \Big) \cos (2\phi - \beta_2) \Big)$$

$$+ \varepsilon_1 \frac{1}{\sin\theta} \frac{d}{d\phi} \Big\{ \Big(\frac{10}{7} P_{2,1} + P_{4,1} \Big) \cos (\phi + \alpha_1 - \beta_2) - P_{4,3} \cos (3\phi - \alpha_1 - \beta_2) \Big\}$$

$$+ \varepsilon_2 \Big\{ - \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(P_{4,4} \cos (4\phi - \alpha_2 - \beta_2) \Big) + \frac{d}{d\theta} \cdot 4 P_1 \sin (\alpha_2 - \beta_2) \Big\} \Big]$$

$$+ na W_2 \Big[\varepsilon \frac{d}{d\theta} \Big(\frac{11}{7} P_2 - \frac{27}{35} P_4 \Big)$$

$$+ \varepsilon_1 \Big\{ \frac{d}{d\theta} \Big(\frac{93}{56} P_{2,1} - \frac{9}{4} P_{4,1} \Big) \cos (\phi - \alpha_1) + \frac{1}{\sin\theta} \frac{d}{d\phi} \Big(- \frac{3}{8} P_{1,1} \sin (\phi - \alpha_1) \Big) \Big\}$$

$$+\varepsilon_2\frac{d}{d\theta}\left(\frac{27}{14}P_{2,2}-\frac{9}{4}P_{4,2}\right)\cos(2\phi-\alpha_2)\Big]$$

$$+naW_{2,1}\Big[\varepsilon\Big\{\frac{d}{d\theta}\Big(-\frac{5}{56}P_{2,1}-\frac{9}{4}P_{4,1}\Big)\cos(\phi-\beta_1)$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big(\frac{3}{8}P_{1,1}\sin(\phi-\beta_1)\Big)\Big\}$$

$$+\varepsilon_1\frac{d}{d\theta}\Big\{\Big(\frac{11}{42}P_2+\frac{6}{35}P_4\Big)\cos(\alpha_1-\beta_1)+\Big(\frac{11}{28}P_{2,2}-\frac{3}{4}P_{4,2}\Big)\cos(2\phi-\alpha_1-\beta_1)\Big\}$$

$$+\varepsilon_2\Big\{\frac{d}{d\theta}\Big(\Big(\frac{39}{56}P_{2,1}+\frac{3}{4}P_{4,1}\Big)\cos(\phi-\alpha_2+\beta_1)-\frac{3}{4}P_{4,3}\cos(3\phi-\alpha_2-\beta_1)\Big)$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big(-\frac{9}{8}P_{1,1}\sin(\phi-\alpha_2+\beta_1)+\frac{7}{12}P_{3,3}\sin(3\phi-\alpha_2-\beta_1)\Big)\Big\}\Big]$$

$$+naW_{2,2}\Big[-\varepsilon\frac{d}{d\theta}\Big(\frac{71}{14}P_{2,2}+\frac{9}{4}P_{4,2}\Big)\cos(2\phi-\beta_2)$$

$$+\varepsilon_1\Big\{\frac{d}{d\theta}\Big(\Big(\frac{137}{56}P_{2,1}+\frac{3}{4}P_{4,1}\Big)\cos(\phi-\beta_2+\alpha_1)-\frac{3}{4}P_{4,3}\cos(3\phi-\alpha_1-\beta_2)\Big)$$

$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\Big(\frac{9}{8}P_{1,1}\sin(\phi-\beta_2+\alpha_1)-\frac{7}{12}P_{3,3}\sin(3\phi-\alpha_1-\beta_2)\Big)\Big\}$$

$$-\varepsilon_2\frac{d}{d\theta}\Big\{\Big(\frac{44}{21}P_2+\frac{6}{35}P_4\Big)\cos(\alpha_2-\beta_2)+\frac{3}{4}P_{4,4}\cos(4\phi-\alpha_2-\beta_2)\Big\}\Big]. \qquad (151)$$

In expanding the right-hand side of this identity use has been made of such of the formulae (7)–(67) as apply to the third surface condition.

§45. In the subsidiary terms of the general solution all harmonic terms with double suffixes in which the first suffix is 3, with the exception of

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\Big\{P_{3,3}\,{\cos\atop\sin}\,\phi\Big\},$$

must be transformed by means of (68)–(72). Also to balance the uncompensated terms we must introduce on the right of the third surface equation

$$\mathrm{a}_3\sin\phi+\mathrm{b}_3\cos\phi+\mathrm{d}_3\cos\theta\sin\phi+\mathrm{e}_3\cos\theta\cos\phi+f_3\sin\theta,$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\{P_{1,1}(-\mathrm{a}_3\cos\phi+\mathrm{b}_3\sin\phi)+P_{2,1}(-\mathrm{d}_3\cos\phi+\mathrm{e}_3\sin\phi)\}-f_3\frac{dP_1}{d\theta}.$$

This being done, we have

$$
\begin{aligned}
&\frac{1}{\sin\theta}\,\frac{d}{d\phi}\Big[(C_2Y_{2,1}+D_2Z_{2,1})\,X'_{2,1}+\{(C_2\delta Y_{2,1}+D_2\delta Z_{2,1},\cos\phi\\
&\quad+(C_2\delta Y'_{2,1}+D_2\delta Z'_{2,1})\sin\phi\}\,P_{2,1}\\
&\quad+(C_2Y_{2,2}+D_2Z_{2,2})\,X'_{2,2}+\{(C_2\delta Y_{2,2}+D_2\delta Z_{2,2})\cos 2\phi\\
&\quad+(C_2\delta Y'_{2,2}+D_2\delta Z'_{2,2})\sin 2\phi\}\,P_{2,2}\Big]\\
&\quad+(C_3Y_{3,1}+D_3Z_{3,1})\Big\{-\frac{1}{2}\,\frac{d}{d\theta}\,(P_{2,1}\sin\phi)+\frac{3}{10}\,\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,(P_{1,1}\cos\phi)\Big\}\\
&\quad+(C_3Y'_{3,1}+D_3Z'_{3,1})\Big\{\frac{1}{2}\,\frac{d}{d\theta}\,(P_{2,1}\cos\phi)+\frac{3}{10}\,\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,(P_{1,1}\sin\phi)\Big\}\\
&\quad+\frac{d}{d\theta}\,[-(C_3Y_{3,2}+D_3Z_{3,2})\,P_{2,2}\sin 2\phi+(C_3Y'_{3,2}+D_3Z'_{3,2})\,P_{2,2}\cos 2\phi]\\
&\quad+\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,[\{(C_3Y_{3,3}+D_3Z_{3,3})\cos 3\phi+(C_3Y'_{3,3}+D_3Z'_{3,3})\sin 3\phi\}\,P_{3,3}\\
&\quad+\{(C_4Y_{4,1}+D_4Z_{4,1})\cos\ \ \phi+(C_4Y'_{4,1}+D_4Z'_{4,1})\sin\ \ \phi\}\,P_{4,1}\\
&\quad+\{(C_4Y_{4,2}+D_4Z_{4,2})\cos 2\phi+(C_4Y'_{4,2}+D_4Z'_{4,2})\sin 2\phi\}\,P_{4,2}\\
&\quad+\{(C_4Y_{4,3}+D_4Z_{4,3})\cos 3\phi+(C_4Y'_{4,3}+D_4Z'_{4,3})\sin 3\phi\}\,P_{4,3}\\
&\quad+\{(C_4Y_{4,4}+D_4Z_{4,4})\cos 4\phi+(C_4Y'_{4,4}+D_4Z'_{4,4})\sin 4\phi\}\,P_{4,4}]\\
&\quad-na\,\frac{d}{d\theta}\,[W_2P_2+W_{2,1}X'_{2,1}+W_{2,2}X'_{2,2}+\delta W_2P_2+(\delta W_{2,1}\cos\phi+\delta W'_{2,1}\sin\phi)\,P_{2,1}\\
&\quad+(\delta W_{2,2}\cos 2\phi+\delta W'_{2,2}\sin 2\phi)\,P_{2,2}]\\
&\quad-2na^2W_3\,\frac{dP_3}{d\theta}\\
&\quad-2na^2\,\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,\Big[\Big(-\frac{32}{35}\,P_{2,1}+3P_{4,1}\Big)(W_{3,1}\sin\phi-W'_{3,1}\cos\phi)\\
&\quad+\Big(-\frac{2}{7}\,P_{2,2}+\frac{3}{2}\,P_{4,2}\Big)(W_{3,2}\sin 2\phi-W'_{3,2}\cos 2\phi)\\
&\quad+P_{4,3}(W_{3,3}\sin 3\phi-W'_{3,3}\cos 3\phi)\Big]\\
&\quad-3na^3\,\frac{d}{d\theta}\,[W_4P_4+(W_{4,1}\cos\phi+W'_{4,1}\sin\phi)\,P_{4,1}\\
&\quad+(W_{4,2}\cos 2\phi+W'_{4,2}\sin 2\phi)\,P_{4,2}+(W_{4,3}\cos 3\phi+W'_{4,3}\sin 3\phi)\,P_{4,3}\\
&\quad+(W_{4,4}\cos 4\phi+W'_{4,4}\sin 4\phi)\,P_{4,4}]\\
&\quad+\text{expression on right-hand side of (151)}\\
&=\frac{1}{\sin\theta}\,\frac{d}{d\phi}\,[(\Theta_{2,1}-M_2V_{2,1})\,X'_{2,1}+(\Theta_{2,2}-M_2V_{2,2})\,X'_{2,2}]
\end{aligned}
$$

$$-\frac{d}{d\theta}\left(\Phi_2 P_2 + \Phi_{2,1} X'_{2,1} + \Phi_{2,2} X'_{2,2}\right)$$
$$+\frac{1}{\sin\theta}\frac{d}{d\phi}\{P_{1,1}(-\mathrm{a}_3\cos\phi + \mathrm{b}_3\sin\phi)$$
$$+P_{2,1}(-\mathrm{d}_3\cos\phi + \mathrm{e}_3\sin\phi)\} - f_3\frac{dP_1}{d\theta}. \tag{152}$$

§46. As in the second surface equation, we shall equate to zero the coefficients of the separate differentials, all the terms being supposed on one side of the equation. For first approximations to Y_2, Z_2, $Y_{2,1}$, $Z_{2,1}$, $Y_{2,2}$, $Z_{2,2}$, $W_{2,1}$, $W_{2,2}$ we should obtain equations which are obviously identical with (114)–(118). For first approximation to W_2 we have

$$W_2 = \frac{1}{na}\Phi_2. \tag{153}$$

For the subsidiary constants the equations obtained are as follows:

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,1}\cos\phi)$,

$$C_2\delta Y_{2,1} + D_2\delta Z_{2,1} = \frac{1}{7}\varepsilon_1\cos\alpha_1(\overline{N} - 9\overline{O}) - \frac{1}{7}\varepsilon\cos\beta_1\left(a\frac{d\overline{M}_1}{da} - 12\overline{O}_1\right)$$
$$+\frac{1}{7}\varepsilon_2\cos(\alpha_2 - \beta_1)\left(2a\frac{d\overline{M}_1}{da} + 4\overline{N}_1 - 4\overline{O}_1\right)$$
$$-\frac{1}{7}\varepsilon_1\cos(\beta_2 - \alpha_1)\left(4a\frac{d\overline{M}_2}{da} + 2\overline{N}_2 - 10\overline{O}_2\right) + \frac{64}{35}na^2 W'_{3,1} - \mathrm{d}_3, \tag{154}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,1}\sin\phi)$,

$$C_2\delta Y'_{2,1} + D_2\delta Z'_{2,1} = \frac{1}{7}\varepsilon_1\sin\alpha_1(\overline{N} - 9\overline{O}) - \frac{1}{7}\varepsilon\sin\beta_1\left(a\frac{d\overline{M}_1}{da} - 12\overline{O}_1\right)$$
$$+\frac{1}{7}\varepsilon_2\sin(\alpha_2 - \beta_1)\left(2a\frac{d\overline{M}_1}{da} + 4\overline{N}_1 - 4\overline{O}_1\right)$$
$$-\frac{1}{7}\varepsilon_1\sin(\beta_2 - \alpha_1)\left(4a\frac{d\overline{M}_2}{da} + 2\overline{N}_2 - 10\overline{O}_2\right) - \frac{64}{35}na^2 W_{3,1} + \mathrm{e}_3, \tag{155}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,2}\cos 2\phi)$,

$$C_2\delta Y_{2,2} + D_2\delta Z_{2,2} = -\frac{1}{7}\varepsilon_2\cos\alpha_2(2\overline{N} + 3\overline{O})$$
$$-\frac{1}{28}\varepsilon_1\cos(\alpha_1 + \beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 - 3\overline{O}_1\right)$$
$$+\frac{1}{7}\varepsilon\cos\beta_2\left(2a\frac{d\overline{M}_2}{da} - 3\overline{O}_2\right) + \frac{4}{7}na^2 W'_{3,2}, \tag{156}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{2,2}\sin 2\phi)$,

$$C_2\delta Y'_{2,2} + D_2\delta Z'_{2,2} = -\frac{1}{7}\varepsilon_2\sin\alpha_2(2\overline{N}+3\overline{O}) - \frac{1}{28}\varepsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 - 3\overline{O}_1\right) + \frac{1}{7}\varepsilon\sin\beta_2\left(2a\frac{d\overline{M}_2}{da} - 3\overline{O}_2\right) - \frac{4}{7}na^2W_{3,2}, \tag{157}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,3}\cos 3\phi)$,

$$C_3Y_{3,3} + D_3Z_{3,3} = \frac{7}{12}na\{\varepsilon_2\sin(\alpha_2+\beta_1)W_{2,1} - \varepsilon_1\sin(\alpha_1+\beta_2)W_{2,2}\}, \tag{158}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{3,3}\sin 3\phi)$,

$$C_3Y'_{3,3} + D_3Z'_{3,3} = \frac{7}{12}na\{-\varepsilon_2\cos(\alpha_2+\beta_1)W_{2,1} + \varepsilon_1\cos(\alpha_1+\beta_2)W_{2,2}\}, \tag{159}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,1}\cos\phi)$,

$$C_4Y_{4,1} + D_4Z_{4,1} = \frac{3}{2}\varepsilon_1\cos\alpha_1(\overline{N}-2\overline{O}) - \frac{3}{2}\varepsilon\cos\beta_1\left(a\frac{d\overline{M}_1}{da} + 2\overline{O}_1\right) - \frac{1}{2}\varepsilon_2\cos(\alpha_2-\beta_1)\left(a\frac{d\overline{M}_1}{da} + 2\overline{N}_1 - 2\overline{O}_1\right) + \frac{1}{2}\varepsilon_1\cos(\beta_2-\alpha_1)\left(2a\frac{d\overline{M}_2}{da} + \overline{N}_2 + 2\overline{O}_2\right) - 6na^2W'_{3,1}, \tag{160}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,1}\sin\phi)$,

$$C_4Y'_{4,1} + D_4Z'_{4,1} = \frac{3}{2}\varepsilon_1\sin\alpha_1(\overline{N}-2\overline{O}) - \frac{3}{2}\varepsilon\sin\beta_1\left(a\frac{d\overline{M}_1}{da} + 2\overline{O}_1\right) - \frac{1}{2}\varepsilon_2\sin(\alpha_2-\beta_1)\left(a\frac{d\overline{M}_1}{da} + 2\overline{N}_1 - 2\overline{O}_1\right) + \frac{1}{2}\varepsilon_1\sin(\beta_2-\alpha_1)\left(2a\frac{d\overline{M}_2}{da} + \overline{N}_2 + 2\overline{O}_2\right) + 6na^2W_{3,1}, \tag{161}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,2}\cos 2\phi)$,

$$C_4Y_{4,2} + D_4Z_{4,2} = \frac{3}{2}\varepsilon_2\cos\alpha_2(\overline{N}-2\overline{O}) - \frac{1}{4}\varepsilon_1\cos(\alpha_1+\beta_1)\left(a\frac{d\overline{M}_1}{da} - \overline{N}_1 + 4\overline{O}_1\right) - \frac{3}{2}\varepsilon\cos\beta_2\left(a\frac{d\overline{M}_2}{da} + 2\overline{O}_2\right) - 3na^2W'_{3,2}, \tag{162}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,2}\sin 2\phi)$,

$$C_4Y'_{4,2}+D_4Z'_{4,2}=\frac{3}{2}\epsilon_2\sin\alpha_2(\overline{N}-2\overline{O})-\frac{1}{4}\epsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\right)$$
$$-\frac{3}{2}\epsilon\sin\beta_2\left(a\frac{d\overline{M}_2}{da}+2\overline{O}_2\right)+3na^2W_{3,2}, \qquad (163)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\cos 3\phi)$,

$$C_4Y_{4,3}+D_4Z_{4,3}=-\frac{1}{6}\epsilon_2\cos(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da}-2\overline{N}_1+6\overline{O}_1\right)$$
$$-\frac{1}{6}\epsilon_1\cos(\alpha_1+\beta_2)\left(2a\frac{d\overline{M}_2}{da}-\overline{N}_2+6\overline{O}_2\right)-2na^2W'_{3,3}, \qquad (164)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,3}\sin 3\phi)$,

$$C_4Y'_{4,3}+D_4Z'_{4,3}=-\frac{1}{6}\epsilon_2\sin(\alpha_2+\beta_1)\left(a\frac{d\overline{M}_1}{da}-2\overline{N}_1+6\overline{O}_1\right)$$
$$-\frac{1}{6}\epsilon_1\sin(\alpha_1+\beta_2)\left(2a\frac{d\overline{M}_2}{da}-\overline{N}_2+6\overline{O}_2\right)+2na^2W_{3,3}, \qquad (165)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,4}\cos 4\phi)$,

$$C_4Y_{4,4}+D_4Z_{4,4}=-\frac{1}{4}\epsilon_2\cos(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \qquad (166)$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{4,4}\sin 4\phi)$,

$$C_4Y'_{4,4}+D_4Z'_{4,4}=-\frac{1}{4}\epsilon_2\sin(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\right), \qquad (167)$$

from coefficient of $\frac{dP_2}{d\theta}$,

$$\delta W_2=\frac{11}{42}\{6\epsilon W_2+\epsilon_1\cos(\alpha_1-\beta_1)W_{2,1}-8\epsilon_2\cos(\alpha_2-\beta_2)W_{2,2}\}, \qquad (168)$$

from coefficient of $\frac{d}{d\theta}(P_{2,1}\cos\phi)$,

$$\delta W_{2,1}=\frac{1}{2na}(C_3Y'_{3,1}+D_3Z'_{3,1})+\frac{93}{56}\epsilon_1\cos\alpha_1 W_2$$
$$+\frac{1}{56}W_{2,1}\{-5\epsilon\cos\beta_1+39\epsilon_2\cos(\alpha_2-\beta_1)\}+\frac{137}{56}\epsilon_1\cos(\beta_2-\alpha_1)W_{2,2}, \qquad (169)$$

from coefficient of $\frac{d}{d\theta}(P_{2,1}\sin\phi)$,

$$\delta W'_{2,1} = -\frac{1}{2na}(C_3Y_{3,1}+D_3Z_{3,1}) + \frac{93}{56}\varepsilon_1\sin\alpha_1 W_2$$
$$+\frac{1}{56}W_{2,1}\{-5\varepsilon\sin\beta_1+39\varepsilon_2\sin(\alpha_2-\beta_1)\}+\frac{137}{56}\varepsilon_1\sin(\beta_2-\alpha_1)W_{2,2}, \quad (170)$$

from coefficient of $\frac{d}{d\theta}(P_{2,2}\cos 2\phi)$,

$$\delta W_{2,2} = \frac{1}{na}(C_3Y'_{3,2}+D_3Z'_{3,2}) + \frac{27}{14}\varepsilon_2\cos\alpha_2 W_2$$
$$+\frac{11}{28}\varepsilon_1\cos(\alpha_1+\beta_1)W_{2,1}-\frac{71}{14}\varepsilon\cos\beta_2 W_{2,2}, \quad (171)$$

from coefficient of $\frac{d}{d\theta}(P_{2,2}\sin 2\phi)$,

$$\delta W'_{2,2} = -\frac{1}{na}(C_3Y_{3,2}+D_3Z_{3,2}) + \frac{27}{14}\varepsilon_2\sin\alpha_2 W_2$$
$$+\frac{11}{28}\varepsilon_1\sin(\alpha_1+\beta_1)W_{2,1}-\frac{71}{14}\varepsilon\sin\beta_2 W_{2,2}, \quad (172)$$

from coefficient of $\frac{dP_3}{d\theta}$,

$$W_3 = \frac{1}{30na^2}\left\{\varepsilon_1\sin(\alpha_1-\beta_1)\left(a\frac{d\bar{M}_1}{da}+\bar{N}_1\right)-2\varepsilon_2\sin(\alpha_2-\beta_2)\left(a\frac{d\bar{M}_2}{da}+\bar{N}_2\right)\right\}, \quad (173)$$

from coefficient of $\frac{dP_4}{d\theta}$,

$$W_4 = -\frac{1}{35a^2}\{9\varepsilon W_2-2\varepsilon_1\cos(\alpha_1-\beta_1)W_{2,1}+2\varepsilon_2\cos(\alpha_2-\beta_2)W_{2,2}\}, \quad (174)$$

from coefficient of $\frac{d}{d\theta}(P_{4,1}\cos\phi)$,

$$W_{4,1} = \frac{1}{4a^2}\{-3\varepsilon_1\cos\alpha_1 W_2-3\varepsilon\cos\beta_1 W_{2,1}$$
$$+\varepsilon_2\cos(\alpha_2-\beta_1)W_{2,1}+\varepsilon_1\cos(\beta_2-\alpha_1)W_{2,2}\}, \quad (175)$$

from coefficient of $\frac{d}{d\theta}(P_{4,1}\sin\phi)$,

$$W'_{4,1} = \frac{1}{4a^2}\{-3\varepsilon_1\sin\alpha_1 W_2-3\varepsilon\sin\beta_1 W_{2,1}$$
$$+\varepsilon_2\sin(\alpha_2-\beta_1)W_{2,1}+\varepsilon_1\sin(\beta_2-\alpha_1)W_{2,2}\}, \quad (176)$$

from coefficient of $\frac{d}{d\theta}(P_{4,2}\cos 2\phi)$,

$$W_{4,2} = -\frac{1}{4a^2}\{3\epsilon_2\cos\alpha_2 W_2 + \epsilon_1\cos(\alpha_1+\beta_1) W_{2,1} + 3\epsilon\cos\beta_2 W_{2,2}\}, \tag{177}$$

from coefficient of $\frac{d}{d\theta}(P_{4,2}\sin 2\phi)$,

$$W'_{4,2} = -\frac{1}{4a^2}\{3\epsilon_2\sin\alpha_2 W_2 + \epsilon_1\sin(\alpha_1+\beta_1) W_{2,1} + 3\epsilon\sin\beta_2 W_{2,2}\}, \tag{178}$$

from coefficient of $\frac{d}{d\theta}(P_{4,3}\cos 3\phi)$,

$$W_{4,3} = -\frac{1}{4a^2}\{\epsilon_2\cos(\alpha_2+\beta_1) W_{2,1} + \epsilon_1\cos(\alpha_1+\beta_2) W_{2,2}\}, \tag{179}$$

from coefficient of $\frac{d}{d\theta}(P_{4,3}\sin 3\phi)$,

$$W'_{4,3} = -\frac{1}{4a^2}\{\epsilon_2\sin(\alpha_2+\beta_1) W_{2,1} + \epsilon_1\sin(\alpha_1+\beta_2) W_{2,2}\}, \tag{180}$$

from coefficient of $\frac{d}{d\theta}(P_{4,4}\cos 4\phi)$,

$$W_{4,4} = -\frac{1}{4a^2}\epsilon_2\cos(\alpha_2+\beta_2) W_{2,2}, \tag{181}$$

from coefficient of $\frac{d}{d\theta}(P_{4,4}\sin 4\phi)$,

$$W'_{4,4} = -\frac{1}{4a^2}\epsilon_2\sin(\alpha_2+\beta_2) W_{2,2}. \tag{182}$$

Finally, coming to the purely uncompensated terms, we get

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{1,1}\cos\phi)$,

$$\begin{aligned}\mathrm{a}_3 = \frac{3}{8} na\,[&-\epsilon_1\sin\alpha_1 W_2 + \epsilon\sin\beta_1 W_{2,1} \\ &+ 3\{-\epsilon_2\sin(\alpha_2-\beta_1) W_{2,1} + \epsilon_1\sin(\beta_2-\alpha_1) W_{2,2}\}] - \frac{3}{10}(C_3 Y_{3,1} + D_3 Z_{3,1}),\end{aligned} \tag{183}$$

from coefficient of $\frac{1}{\sin\theta}\frac{d}{d\phi}(P_{1,1}\sin\phi)$,

$$\mathrm{b}_3 = \frac{3}{8} na\,[-\epsilon_1\cos\alpha_1 W_2 + \epsilon\cos\beta_1 W_{2,1}$$

$$+3\{-\epsilon_2\cos(\alpha_2-\beta_1)W_{2,1}+\epsilon_1\cos(\beta_2-\alpha_1)W_{2,2}\}]+\frac{3}{10}(C_3Y'_{3,1}+D_3Z'_{3,1}), \quad (184)$$

from coefficient of $\frac{dP_1}{d\theta}$,

$$\mathrm{f}_3 = -\frac{1}{10}\epsilon_1\sin(\alpha_1-\beta_1)\left(a\frac{d\overline{M}_1}{da}+\overline{N}_1-5\overline{O}_1\right)$$

$$-\frac{4}{5}\epsilon_2\sin(\alpha_2-\beta_2)\left(a\frac{d\overline{M}_2}{da}+\overline{N}_2-5\overline{O}_2\right). \quad (185)$$

§47. Of the equations supplied by the second and third surface conditions, some, e. g. (132) and (166), are obviously identical as they stand. Taking either of the equations in such a case along with the equation for the same constants supplied by the first surface condition, we clearly determine the constants without ambiguity. Thus from (107) and (132) we determine at once $Y_{4,4}$ and $Z_{4,4}$.

In the majority of instances, however, whilst corresponding equations from the second and third surface conditions are easily recognizable by means of the suffixes employed, they are not identical as they stand, one containing not infrequently an arbitrary constant which does not appear in the other. To go into all the details of the comparison of the two sets of equations would occupy too much space. A general outline will, it is hoped, suffice.

It will be noticed that $W'_{3,1}$ occurs in (120), (154) and (160); also that (120) and (154) are equations for $C_2\delta Y_{2,1}+D_2\delta Z_{2,1}$, while (160) and (126) are equations for $C_4Y_{4,1}+D_4Z_{4,1}$. We thus take these four equations together.

Equating the values supplied for $C_2\delta Y_{2,1}+D_2\delta Z_{2,1}$ by (120) and (154), we get a simple equation in which the unknowns are d_3 and $W'_{3,1}$; while equating the values supplied for $C_4Y_{4,1}+D_4Z_{4,1}$ by (126) and (160), we obtain the value of $W'_{3,1}$, and so by the previous result the value of d_3 also. We are thus left with two equations from which $W'_{3,1}$ and d_3 have been eliminated. Of these, one taken along with (89) determines $\delta Y_{2,1}$ and $\delta Z_{2,1}$, while the other taken along with (101) determines $Y_{4,1}$ and $Z_{4,1}$.

Similarly we consider (121), (127), (155) and (161) together, determining from them *inter alia* the value of e_3.

In some cases we seem at first sight to have too many equations. For instance, a value for $W'_{3,2}$ can be deduced from (128) and (162) as well as from (122) and (156). The values so found, however, are identical. A similar result holds in the case of the pairs (123), (157) and (129), (163), either of which gives the value of $W_{3,2}$.

Again, $W'_{3,3}$ is given explicitly by (139); but this value is identical with that obtained by eliminating $C_4 Y_{4,3} + D_4 Z_{4,3}$ between (130) and (164). There is a similar identity between the value obtained from (131) and (165) for $W_{3,3}$ and that given explicitly in (138).

The first approximations (117), (118), (153) for $W_{2,1}$, $W_{2,2}$ and W_2 are to be substituted in all terms on the right-hand sides of the equations as they occur multiplied by ε, ε_1 or ε_2. Thus (168) combined with (153) gives the complete value of W_2 as far as the second approximation.

Equating the values supplied for $\delta W_{2,1}$ by (134) and (169) we obtain a simple equation whose unknowns are b_2 and $C_3 Y'_{3,1} + D_3 Z'_{3,1}$. The value of the latter quantity is found by eliminating $W_{4,1}$ between (140) and (175). We thus find the values of b_2 and of $C_3 Y'_{3,1} + D_3 Z'_{3,1}$, and combining the latter result with (95), we determine the values of $Y'_{3,1}$ and $Z'_{3,1}$. Similarly from (135), (170), (141) and (176) we obtain the values of a_2, $\delta W'_{2,1}$, $W'_{4,1}$ and $C_3 Y_{3,1} + D_3 Z_{3,1}$; and taking the last result along with (94), we find $Y_{3,1}$ and $Z_{3,1}$.

Coincident values for $C_3 Y'_{3,2} + D_3 Z'_{3,2}$ are obtainable from the pairs (136), (171) and (142), (177); while coincident values for $C_3 Y_{3,2} + D_3 Z_{3,2}$ are got from the pairs (137), (172) and (143), (178).

Between (144) and (179) we obtain the values of $W_{4,3}$ and $C_3 Y'_{3,3} + D_3 Z'_{3,3}$, the latter value being identical with that given explicitly in (159). Between (145) and (180) we obtain the values of $W'_{4,3}$ and $C_3 Y_{3,3} + D_3 Z_{3,3}$, the latter being identical with (158). The pairs (146), (181) and (147), (182) are identical as they stand.

If the equations are examined in detail it will be found there are enough to determine all the constants, and that any apparently superabundant equations are never inconsistent with but merely combinations of previous equations.

As regards the arbitrary constants appearing in the uncompensated terms, the three, a_1, b_1, c_1 occurring in the first surface condition are given by (109), (110) and (111), when we substitute for the W's their first approximations from (117), (118) and (153). We have already shown how d_3, e_3, a_2, b_2 are found. Of the others, c_2, e_2, d_2 are given by (148), (149) and (150), when we substitute their first approximation values for $W_{2,1}$ and $W_{2,2}$, and replace $W_{3,1}$ and $W'_{3,1}$ by their values determined as explained above. The final three, a_3, b_3, f_3, are given by (183), (184) and (185) when we substitute for the W's their first approximation values, and employ for $C_3 Y'_{3,1} + D_3 Z'_{3,1}$ and $C_3 Y_{3,1} + D_3 Z_{3,1}$ the values whose deduction from (140), (175) and (141), (176) has been already explained.

§48. The values of these constants contained in the uncompensated terms may conveniently be recorded at this stage. They are as follows:

$$\left.\begin{aligned}
a_1 &= \frac{1}{5}\big[-3\varepsilon_1 \sin\alpha_1 \Phi_2 + \{3\varepsilon \sin\beta_1 - 2\varepsilon_2 \sin(\alpha_2 - \beta_1)\}\,\Phi_{2,1} \\
&\qquad + 2\varepsilon_1 \sin(\beta_2 - \alpha_1)\,\Phi_{2,2}\big], \\
b_1 &= \frac{1}{5}\big[3\varepsilon_1 \cos\alpha_1 \Phi_2 - \{3\varepsilon \cos\beta_1 - 2\varepsilon_2 \cos(\alpha_2 - \beta_1)\}\,\Phi_{2,1} \\
&\qquad - 2\varepsilon_1 \cos(\beta_2 - \alpha_1)\,\Phi_{2,2}\big], \\
c_1 &= \frac{1}{5}\big[\varepsilon_1 \sin(\alpha_1 - \beta_1)\,\Phi_{2,1} + 8\varepsilon_2 \sin(\alpha_2 - \beta_2)\,\Phi_{2,2}\big], \\
a_2 &= \frac{3}{10}\big[3\varepsilon_1 \sin\alpha_1 \Phi_2 - \{3\varepsilon \sin\beta_1 - 2\varepsilon_2 \sin(\alpha_2 - \beta_1)\}\,\Phi_{2,1} \\
&\qquad - 2\varepsilon_1 \sin(\beta_2 - \alpha_1)\,\Phi_{2,2}\big], \\
b_2 &= \frac{3}{10}\big[-3\varepsilon_1 \cos\alpha_1 \Phi_2 + \{3\varepsilon \cos\beta_1 - 2\varepsilon_2 \cos(\alpha_2 - \beta_1)\}\,\Phi_{2,1} \\
&\qquad + 2\varepsilon_1 \cos(\beta_2 - \alpha_1)\,\Phi_{2,2}\big], \\
c_2 &= \frac{3}{10}\big[\varepsilon_1 \sin(\alpha_1 - \beta_1)\,\Phi_{2,1} + 8\varepsilon_2 \sin(\alpha_2 - \beta_2)\,\Phi_{2,2}\big], \\
d_2 &= \frac{1}{10}\Big[-3\varepsilon_1 \cos\alpha_1 \Big(a\frac{d\overline{M}}{da} + \overline{N} - 5\overline{O}\Big) \\
&\qquad + \{3\varepsilon \cos\beta_1 - 2\varepsilon_2 \cos(\alpha_2 - \beta_1)\}\Big(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 5\overline{O}_1\Big) \\
&\qquad + 2\varepsilon_1 \cos(\beta_2 - \alpha_1)\Big(a\frac{d\overline{M}_2}{da} + \overline{N}_2 - 5\overline{O}_2\Big)\Big], \\
e_2 &= \frac{1}{10}\Big[-3\varepsilon_1 \sin\alpha_1 \Big(a\frac{d\overline{M}}{da} + \overline{N} - 5\overline{O}\Big) \\
&\qquad + \{3\varepsilon \sin\beta_1 - 2\varepsilon_2 \sin(\alpha_2 - \beta_1)\}\Big(a\frac{d\overline{M}_1}{da} + \overline{N}_1 - 5\overline{O}_1\Big) \\
&\qquad + 2\varepsilon_1 \sin(\beta_2 - \alpha_1)\Big(a\frac{d\overline{M}_2}{da} + \overline{N}_2 - 5\overline{O}_2\Big)\Big],
\end{aligned}\right\} \quad (186)$$

$$\left.\begin{aligned}
\mathrm{a}_3 &= \frac{3}{10}\big[-3\epsilon_1 \sin\alpha_1\Phi_2 + \{3\epsilon\sin\beta_1 - 2\epsilon_2\sin(\alpha_2-\beta_1)\}\Phi_{2,1} \\
&\qquad + 2\epsilon_1\sin(\beta_2-\alpha_1)\Phi_{2,2}\big], \\
\mathrm{b}_3 &= \frac{3}{10}\big[-3\epsilon_1 \cos\alpha_1\Phi_2 + \{3\epsilon\cos\beta_1 - 2\epsilon_2\cos(\alpha_2-\beta_1)\}\Phi_{2,1} \\
&\qquad + 2\epsilon_1\cos(\beta_2-\alpha_1)\Phi_{2,2}\big], \\
\mathrm{d}_3 &= \frac{1}{10}\Big[3\epsilon_1\cos\alpha_1\Big(a\frac{d\overline{M}}{da}+\overline{N}-5\overline{O}\Big) \\
&\qquad -\{3\epsilon\cos\beta_1 - 2\epsilon_2\cos(\alpha_2-\beta_1)\}\Big(a\frac{d\overline{M}_1}{da}+\overline{N}_1-5\overline{O}_1\Big) \\
&\qquad - 2\epsilon_1\cos(\beta_2-\alpha_1)\Big(a\frac{d\overline{M}_2}{da}+\overline{N}_2-5\overline{O}_2\Big)\Big], \\
\mathrm{e}_3 &= \frac{1}{10}\Big[-3\epsilon_1\sin\alpha_1\Big(a\frac{d\overline{M}}{da}+\overline{N}-5\overline{O}\Big) \\
&\qquad +\{3\epsilon\sin\beta_1 - 2\epsilon_2\sin(\alpha_2-\beta_1)\}\Big(a\frac{d\overline{M}_1}{da}+\overline{N}_1-5\overline{O}_1\Big) \\
&\qquad + 2\epsilon_1\sin(\beta_2-\alpha_1)\Big(a\frac{d\overline{M}_2}{da}+\overline{N}_2-5\overline{O}_2\Big)\Big], \\
\mathrm{f}_3 &= -\frac{1}{10}\Big[\epsilon_1\sin(\alpha_1-\beta_1)\Big(a\frac{d\overline{M}_1}{da}+\overline{N}_1-5\overline{O}_1\Big) \\
&\qquad + 8\epsilon_2\sin(\alpha_2-\beta_2)\Big(a\frac{d\overline{M}_2}{da}+\overline{N}_2-5\overline{O}_2\Big)\Big].
\end{aligned}\right\} \quad (186)$$

The existence of the uncompensated terms may seem rather a stumbling block to the reader, but he will presently find that we can give a good account of them.

§49. Before doing this, however, it seems best to collect the results for the arbitrary constants answering to the true elastic strains.

Our previous work has determined Y_0 and all the W constants, and it furnishes pairs of simple equations sufficient to determine all the Y and Z constants. Some of these equations have already appeared in their final form, but it is convenient to record them all together, so that corresponding pairs may be found side by side. After our previous remarks it is unnecessary to state the sources of the several equations. Those bracketed together contain the same pair of constants. We have

$$\begin{aligned}
A_0Y_0 = &-\frac{1}{5}\epsilon\Big(a\frac{d\overline{L}}{da}-6\overline{M}\Big) - \frac{1}{15}\epsilon_1\cos(\alpha_1-\beta_1)\Big(a\frac{d\overline{L}_1}{da}-6\overline{M}_1\Big) \\
&-\frac{4}{15}\epsilon_2\cos(\alpha_2-\beta_2)\Big(a\frac{d\overline{L}_2}{da}-6\overline{M}_2\Big),
\end{aligned} \qquad (187)$$

$$\left.\begin{aligned}
A_2\delta Y_2+B_2\delta Z_2 = {} & -\frac{2}{7}\,\epsilon\left(a\frac{d\overline{L}}{da}-3\overline{M}\right)-\frac{1}{21}\,\epsilon_1\cos(\alpha_1-\beta_1)\left(a\frac{d\overline{L}_1}{da}-3\overline{M}_1\right)\\
& +\frac{8}{21}\,\epsilon_2\cos(\alpha_2-\beta_2)\left(a\frac{d\overline{L}_2}{da}-3\overline{M}_2\right),\\
C_2\delta Y_2+D_2\delta Z_2 = {} & -\frac{1}{7}\,\epsilon\left(a\frac{d\overline{M}}{da}-\overline{N}-3\overline{O}\right)\\
& -\frac{1}{42}\,\epsilon_1\cos(\alpha_1-\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1-3\overline{O}_1\right)\\
& +\frac{4}{21}\,\epsilon_2\cos(\alpha_2-\beta_2)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2-3\overline{O}_2\right),
\end{aligned}\right\}\quad(188)$$

$$\left.\begin{aligned}
A_2\delta Y_{2,1}+B_2\delta Z_{2,1} = {} & -\frac{1}{7}\,\epsilon_1\cos\alpha_1\left(a\frac{d\overline{L}}{da}-3\overline{M}\right)\\
& -\frac{1}{7}\,\epsilon\cos\beta_1\left(a\frac{d\overline{L}_1}{da}-3\overline{M}_1\right)-\frac{2}{7}\,\epsilon_2\cos(\alpha_2-\beta_1)\left(a\frac{d\overline{L}_1}{da}-3\overline{M}_1\right)\\
& -\frac{2}{7}\,\epsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\overline{L}_2}{da}-3\overline{M}_2\right),\\
C_2\delta Y_{2,1}+D_2\delta Z_{2,1} = {} & -\frac{1}{14}\,\epsilon_1\cos\alpha_1\left(a\frac{d\overline{M}}{da}-\overline{N}-3\overline{O}\right)\\
& -\frac{1}{14}\,\epsilon\cos\beta_1\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1-3\overline{O}_1\right)\\
& -\frac{1}{7}\,\epsilon_2\cos(\alpha_2-\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1-3\overline{O}_1\right)\\
& -\frac{1}{7}\,\epsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2-3\overline{O}_2\right),
\end{aligned}\right\}\quad(189)$$

$$\left.\begin{aligned}
A_2\delta Y'_{2,1}+B_2\delta Z'_{2,1} = {} & -\frac{1}{7}\,\epsilon_1\sin\alpha_1\left(a\frac{d\overline{L}}{da}-3\overline{M}\right)\\
& -\frac{1}{7}\,\epsilon\sin\beta_1\left(a\frac{d\overline{L}_1}{da}-3\overline{M}_1\right)-\frac{2}{7}\,\epsilon_2\sin(\alpha_2-\beta_1)\left(a\frac{d\overline{L}_1}{da}-3\overline{M}_1\right)\\
& -\frac{2}{7}\,\epsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\overline{L}_2}{da}-3\overline{M}_2\right),\\
C_2\delta Y'_{2,1}+D_2\delta Z'_{2,1} = {} & -\frac{1}{14}\,\epsilon_1\sin\alpha_1\left(a\frac{d\overline{M}}{da}-\overline{N}-3\overline{O}\right)\\
& -\frac{1}{14}\,\epsilon\sin\beta_1\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1-3\overline{O}_1\right)\\
& -\frac{1}{7}\,\epsilon_2\sin(\alpha_2-\beta_1)\left(a\frac{d\overline{M}_1}{da}-\overline{N}_1-3\overline{O}_1\right)\\
& -\frac{1}{7}\,\epsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da}-\overline{N}_2-3\overline{O}_2\right),
\end{aligned}\right\}\quad(190)$$

$$\left.\begin{aligned}
A_2\delta Y_{2,2}+B_2\delta Z_{2,2} &= \frac{2}{7}\,\epsilon_2\cos\alpha_2\left(a\frac{d\bar{L}}{da}-3\bar{M}\right)\\
&\quad-\frac{1}{14}\,\epsilon_1\cos(\alpha_1+\beta_1)\left(a\frac{d\bar{L}_1}{da}-3\bar{M}_1\right)+\frac{2}{7}\,\epsilon\cos\beta_2\left(a\frac{d\bar{L}_2}{da}-3\bar{M}_2\right),\\
C_2\delta Y_{2,2}+D_2\delta Z_{2,2} &= \frac{1}{7}\,\epsilon_2\cos\alpha_2\left(a\frac{d\bar{M}}{da}-\bar{N}-3\bar{O}\right)\\
&\quad-\frac{1}{28}\,\epsilon_1\cos(\alpha_1+\beta_1)\left(a\frac{d\bar{M}_1}{da}-\bar{N}_1-3\bar{O}_1\right)\\
&\quad+\frac{1}{7}\,\epsilon\cos\beta_2\left(a\frac{d\bar{M}_2}{da}-\bar{N}_2-3\bar{O}_2\right),
\end{aligned}\right\}\quad(191)$$

$$\left.\begin{aligned}
A_2\delta Y'_{2,2}+B_2\delta Z'_{2,2} &= \frac{2}{7}\,\epsilon_2\sin\alpha_2\left(a\frac{d\bar{L}}{da}-3\bar{M}\right)\\
&\quad-\frac{1}{14}\,\epsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\bar{L}_1}{da}-3\bar{M}_1\right)+\frac{2}{7}\,\epsilon\sin\beta_2\left(a\frac{d\bar{L}_2}{da}-3\bar{M}_2\right),\\
C_2\delta Y'_{2,2}+D_2\delta Z'_{2,2} &= \frac{1}{7}\,\epsilon_2\sin\alpha_2\left(a\frac{d\bar{M}}{da}-\bar{N}-3\bar{O}\right)\\
&\quad-\frac{1}{28}\,\epsilon_1\sin(\alpha_1+\beta_1)\left(a\frac{d\bar{M}_1}{da}-\bar{N}_1-3\bar{O}_1\right)\\
&\quad+\frac{1}{7}\,\epsilon\sin\beta_2\left(a\frac{d\bar{M}_2}{da}-\bar{N}_2-3\bar{O}_2\right),
\end{aligned}\right\}\quad(192)$$

$$\begin{aligned}\frac{A_3Y_3+B_3Z_3}{12}=\frac{C_3Y_3+D_3Z_3}{7}&=\frac{1}{15}\{-\epsilon_1\sin(\alpha_1-\beta_1)\,\Phi_{2,1}\\&\quad+2\epsilon_2\sin(\alpha_2-\beta_2)\,\Phi_{2,2}\},\end{aligned}\quad(193)$$

$$\begin{aligned}\frac{A_3Y_{3,1}+B_3Z_{3,1}}{12}=\frac{C_3Y_{3,1}+D_3Z_{3,1}}{7}&=\frac{1}{4}\,[\epsilon_1\sin\alpha_1\Phi_2\\-\{\epsilon\sin\beta_1+\epsilon_2\sin(\alpha_2-\beta_1)\}\,\Phi_{2,1}&+\epsilon_1\sin(\beta_2-\alpha_1)\,\Phi_{2,2}],\end{aligned}\quad(194)$$

$$\begin{aligned}\frac{A_3Y'_{3,1}+B_3Z'_{3,1}}{12}=\frac{C_3Y'_{3,1}+D_3Z'_{3,1}}{7}&=\frac{1}{4}\,[-\epsilon_1\cos\alpha_1\Phi_2\\+\{\epsilon\cos\beta_1+\epsilon_2\cos(\alpha_2-\beta_1)\}\,\Phi_{2,1}&-\epsilon_1\cos(\beta_2-\alpha_1)\,\Phi_{2,2}],\end{aligned}\quad(195)$$

$$\frac{A_3Y_{3,2}+B_3Z_{3,2}}{12}=\frac{C_3Y_{3,2}+D_3Z_{3,2}}{7}=\frac{1}{2}\{\epsilon_2\sin\alpha_2\Phi_2-\epsilon\sin\beta_2\Phi_{2,2}\},\quad(196)$$

$$\frac{A_3Y'_{3,2}+B_3Z'_{3,2}}{12}=\frac{C_3Y'_{3,2}+D_3Z'_{3,2}}{7}=\frac{1}{2}\{-\epsilon_2\cos\alpha_2\Phi_2+\epsilon\cos\beta_2\Phi_{2,2}\},\quad(197)$$

$$\begin{aligned}\frac{A_3Y_{3,3}+B_3Z_{3,3}}{12}=\frac{C_3Y_{3,3}+D_3Z_{3,3}}{7}&=\frac{1}{12}\{\epsilon_2\sin(\alpha_2+\beta_1)\,\Phi_{2,1}\\&\quad-\epsilon_1\sin(\beta_2+\alpha_1)\,\Phi_{2,2}\},\end{aligned}\quad(198)$$

$$\begin{aligned}\frac{A_3Y'_{3,3}+B_3Z'_{3,3}}{12}=\frac{C_3Y'_{3,3}+D_3Z'_{3,3}}{7}&=\frac{1}{12}\{-\epsilon_2\cos(\alpha_2+\beta_1)\,\Phi_{2,1}\\&\quad+\epsilon_1\cos(\beta_2+\alpha_1)\,\Phi_{2,2}\},\end{aligned}\quad(199)$$

$$\left.\begin{aligned}
A_4Y_4+B_4Z_4={}&-\frac{18}{35}\varepsilon\left(a\frac{d\bar{L}}{da}+4\bar{M}\right)+\frac{4}{35}\varepsilon_1\cos(\alpha_1-\beta_1)\left(a\frac{d\bar{L}_1}{da}+4\bar{M}_1\right)\\
&-\frac{4}{35}\varepsilon_2\cos(\alpha_2-\beta_2)\left(a\frac{d\bar{L}_2}{da}+4\bar{M}_2\right),\\
C_4Y_4+D_4Z_4={}&-\frac{9}{35}\varepsilon\left(a\frac{d\bar{M}}{da}-\bar{N}+4\bar{O}\right)\\
&+\frac{2}{35}\varepsilon_1\cos(\alpha_1-\beta_1)\left(a\frac{d\bar{M}_1}{da}-\bar{N}_1+4\bar{O}_1\right)\\
&-\frac{2}{35}\varepsilon_2\cos(\alpha_2-\beta_2)\left(a\frac{d\bar{M}_2}{da}-\bar{N}_2+4\bar{O}_2\right),
\end{aligned}\right\}\quad(200)$$

$$\left.\begin{aligned}
A_4Y_{4,1}+B_4Z_{4,1}={}&-\frac{3}{2}\varepsilon_1\cos\alpha_1\left(a\frac{d\bar{L}}{da}+4\bar{M}\right)\\
&-\frac{1}{2}\{3\varepsilon\cos\beta_1-\varepsilon_2\cos(\alpha_2-\beta_1)\}\left(a\frac{d\bar{L}_1}{da}+4\bar{M}_1\right)\\
&+\frac{1}{2}\varepsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\bar{L}_2}{da}+4\bar{M}_2\right),\\
C_4Y_{4,1}+D_4Z_{4,1}={}&-\frac{3}{4}\varepsilon_1\cos\alpha_1\left(a\frac{d\bar{M}}{da}-\bar{N}+4\bar{O}\right)\\
&-\frac{1}{4}\{3\varepsilon\cos\beta_1-\varepsilon_2\cos(\alpha_2-\beta_1)\}\left(a\frac{d\bar{M}_1}{da}-\bar{N}_1+4\bar{O}_1\right)\\
&+\frac{1}{4}\varepsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\bar{M}_2}{da}-\bar{N}_2+4\bar{O}_2\right),
\end{aligned}\right\}\quad(201)$$

$$\left.\begin{aligned}
A_4Y'_{4,1}+B_4Z'_{4,1}={}&-\frac{3}{2}\varepsilon_1\sin\alpha_1\left(a\frac{d\bar{L}}{da}+4\bar{M}\right)\\
&-\frac{1}{2}\{3\varepsilon\sin\beta_1-\varepsilon_2\sin(\alpha_2-\beta_1)\}\left(a\frac{d\bar{L}_1}{da}+4\bar{M}_1\right)\\
&+\frac{1}{2}\varepsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\bar{L}_2}{da}+4\bar{M}_2\right),\\
C_4Y'_{4,1}+D_4Z'_{4,1}={}&-\frac{3}{4}\varepsilon_1\sin\alpha_1\left(a\frac{d\bar{M}}{da}-\bar{N}+4\bar{O}\right)\\
&-\frac{1}{4}\{3\varepsilon\sin\beta_1-\varepsilon_2\sin(\alpha_2-\beta_1)\}\left(a\frac{d\bar{M}_1}{da}-\bar{N}_1+4\bar{O}_1\right)\\
&+\frac{1}{4}\varepsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\bar{M}_2}{da}-\bar{N}_2+4\bar{O}_2\right),
\end{aligned}\right\}\quad(202)$$

$$\left.\begin{aligned}
A_4Y_{4,2}+B_4Z_{4,2}=&-\frac{1}{2}\Big\{3\varepsilon_2\cos\alpha_2\Big(a\frac{d\overline{L}}{da}+4\overline{M}\Big)\\
&+\varepsilon_1\cos(\alpha_1+\beta_1)\Big(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\Big)+3\varepsilon\cos\beta_2\Big(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\Big)\Big\},\\
C_4Y_{4,2}+D_4Z_{4,2}=&-\frac{1}{4}\Big\{3\varepsilon_2\cos\alpha_2\Big(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\Big)\\
&+\varepsilon_1\cos(\alpha_1+\beta_1)\Big(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\Big)\\
&+3\varepsilon\cos\beta_2\Big(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\Big)\Big\},
\end{aligned}\right\}\quad(203)$$

$$\left.\begin{aligned}
A_4Y'_{4,2}+B_4Z'_{4,2}=&-\frac{1}{2}\Big[3\varepsilon_2\sin\alpha_2\Big(a\frac{d\overline{L}}{da}+4\overline{M}\Big)\\
&+\varepsilon_1\sin(\alpha_1+\beta_1)\Big(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\Big)+3\varepsilon\sin\beta_2\Big(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\Big)\Big],\\
C_4Y'_{4,2}+D_4Z'_{4,2}=&-\frac{1}{4}\Big[3\varepsilon_2\sin\alpha_2\Big(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\Big)\\
&+\varepsilon_1\sin(\alpha_1+\beta_1)\Big(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\Big)\\
&+3\varepsilon\sin\beta_2\Big(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\Big)\Big],
\end{aligned}\right\}\quad(204)$$

$$\left.\begin{aligned}
A_4Y_{4,3}+B_4Z_{4,3}=&-\frac{1}{2}\Big[\varepsilon_2\cos(\alpha_2+\beta_1)\Big(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\Big)\\
&+\varepsilon_1\cos(\beta_2+\alpha_1)\Big(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\Big)\Big],\\
C_4Y_{4,3}+D_4Z_{4,3}=&-\frac{1}{4}\Big[\varepsilon_2\cos(\alpha_2+\beta_1)\Big(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\Big)\\
&+\varepsilon_1\cos(\beta_2+\alpha_1)\Big(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\Big)\Big],
\end{aligned}\right\}\quad(205)$$

$$\left.\begin{aligned}
A_4Y'_{4,3}+B_4Z'_{4,3}=&-\frac{1}{2}\Big[\varepsilon_2\sin(\alpha_2+\beta_1)\Big(a\frac{d\overline{L}_1}{da}+4\overline{M}_1\Big)\\
&+\varepsilon_1\sin(\beta_2+\alpha_1)\Big(a\frac{d\overline{L}_2}{da}+4\overline{M}_2\Big)\Big],\\
C_4Y'_{4,3}+D_4Z'_{4,3}=&-\frac{1}{4}\Big[\varepsilon_2\sin(\alpha_2+\beta_1)\Big(a\frac{d\overline{M}_1}{da}-\overline{N}_1+4\overline{O}_1\Big)\\
&+\varepsilon_1\sin(\beta_2+\alpha_1)\Big(a\frac{d\overline{M}_2}{da}-\overline{N}_2+4\overline{O}_2\Big)\Big],
\end{aligned}\right\}\quad(206)$$

$$\left.\begin{aligned}A_4 Y_{4,4} + B_4 Z_{4,4} &= -\frac{1}{2}\varepsilon_2 \cos(\alpha_2+\beta_2)\left(a\frac{d\overline{L}_2}{da} + 4\overline{M}_2\right),\\ C_4 Y_{4,4} + D_4 Z_{4,4} &= -\frac{1}{4}\varepsilon_2 \cos(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_2\right),\end{aligned}\right\} \quad (207)$$

$$\left.\begin{aligned}A_4 Y'_{4,4} + B_4 Z'_{4,4} &= -\frac{1}{2}\varepsilon_2 \sin(\alpha_2+\beta_2)\left(a\frac{d\overline{L}_2}{da} + 4\overline{M}_2\right),\\ C_4 Y'_{4,4} + D_4 Z'_{4,4} &= -\frac{1}{4}\varepsilon_2 \sin(\alpha_2+\beta_2)\left(a\frac{d\overline{M}_2}{da} - \overline{N}_2 + 4\overline{O}_2\right),\end{aligned}\right\} \quad (208)$$

$$\delta W_2 = \frac{11}{42na}\left[6\varepsilon\Phi_2 + \varepsilon_1\cos(\alpha_1-\beta_1)\Phi_{2,1} - 8\varepsilon_2\cos(\alpha_2-\beta_2)\Phi_{2,2}\right], \quad (209)$$

$$\delta W_{2,1} = \frac{11}{14na}\left[\varepsilon_1\cos\alpha_1\Phi_2 + \{\varepsilon\cos\beta_1 + 2\varepsilon_2\cos(\alpha_2-\beta_1)\}\Phi_{2,1} + 2\varepsilon_1\cos(\beta_2-\alpha_1)\Phi_{2,2}\right], \quad (210)$$

$$\delta W'_{2,1} = \frac{11}{14na}\left[\varepsilon_1\sin\alpha_1\Phi_2 + \{\varepsilon\sin\beta_1 + 2\varepsilon_2\sin(\alpha_2-\beta_1)\}\Phi_{2,1} + 2\varepsilon_1\sin(\beta_2-\alpha_1)\Phi_{2,2}\right], \quad (211)$$

$$\delta W_{2,2} = -\frac{11}{28na}\left[4\varepsilon_2\cos\alpha_2\Phi_2 - \varepsilon_1\cos(\alpha_1+\beta_1)\Phi_{2,1} + 4\varepsilon\cos\beta_2\Phi_{2,2}\right], \quad (212)$$

$$\delta W'_{2,2} = -\frac{11}{28na}\left[4\varepsilon_2\sin\alpha_2\Phi_2 - \varepsilon_1\sin(\alpha_1+\beta_1)\Phi_{2,1} + 4\varepsilon\sin\beta_2\Phi_{2,2}\right], \quad (213)$$

$$W_3 = \frac{1}{30na^2}\left[\varepsilon_1\sin(\alpha_1-\beta_1)\left(a\frac{d\overline{M}_1}{da} + \overline{N}_1\right) - 2\varepsilon_2\sin(\alpha_2-\beta_2)\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2\right)\right], \quad (214)$$

$$W_{3,1} = \frac{1}{8na^2}\left[-\varepsilon_1\sin\alpha_1\left(a\frac{d\overline{M}}{da} + \overline{N}\right) + \{\varepsilon\sin\beta_1 + \varepsilon_2\sin(\alpha_2-\beta_1)\}\left(a\frac{d\overline{M}_1}{da} + \overline{N}_1\right) - \varepsilon_1\sin(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2\right)\right], \quad (215)$$

$$W'_{3,1} = \frac{1}{8na^2}\left[\varepsilon_1\cos\alpha_1\left(a\frac{d\overline{M}}{da} + \overline{N}\right) - \{\varepsilon\cos\beta_1 + \varepsilon_2\cos(\alpha_2-\beta_1)\}\left(a\frac{d\overline{M}_1}{da} + \overline{N}_1\right) + \varepsilon_1\cos(\beta_2-\alpha_1)\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2\right)\right], \quad (216)$$

$$W_{3,2} = -\frac{1}{4na^2}\left[\varepsilon_2\sin\alpha_2\left(a\frac{d\overline{M}}{da} + \overline{N}\right) - \varepsilon\sin\beta_2\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2\right)\right], \quad (217)$$

$$W'_{3,2} = \frac{1}{4na^2}\left[\varepsilon_2\cos\alpha_2\left(a\frac{d\overline{M}}{da} + \overline{N}\right) - \varepsilon\cos\beta_2\left(a\frac{d\overline{M}_2}{da} + \overline{N}_2\right)\right], \quad (218)$$

$$W_{3,3} = -\frac{1}{24na^2}\Big[\varepsilon_2 \sin(\alpha_2+\beta_1)\Big(a\frac{d\overline{M}_1}{da}+\overline{N}_1\Big) - \varepsilon_1 \sin(\beta_2+\alpha_1)\Big(a\frac{d\overline{M}_2}{da}+\overline{N}_2\Big)\Big], \tag{219}$$

$$W'_{3,3} = \frac{1}{24na^2}\Big[\varepsilon_2 \cos(\alpha_2+\beta_1)\Big(a\frac{d\overline{M}_1}{da}+\overline{N}_1\Big) - \varepsilon_1 \cos(\beta_2+\alpha_1)\Big(a\frac{d\overline{M}_2}{da}+\overline{N}_2\Big)\Big], \tag{220}$$

$$W_4 = -\frac{1}{35na^2}\left[9\varepsilon\Phi_2 - 2\varepsilon_1\cos(\alpha_1-\beta_1)\,\Phi_{2,1} + 2\varepsilon_2\cos(\alpha_2-\beta_2)\,\Phi_{2,2}\right], \tag{221}$$

$$W_{4,1} = -\frac{1}{4na^2}\left[3\varepsilon_1\cos\alpha_1\Phi_2 + \{3\varepsilon\cos\beta_1 - \varepsilon_2\cos(\alpha_2-\beta_1)\}\,\Phi_{2,1} - \varepsilon_1\cos(\beta_2-\alpha_1)\,\Phi_{2,2}\right], \tag{222}$$

$$W'_{4,1} = -\frac{1}{4na^2}\left[3\varepsilon_1\sin\alpha_1\Phi_2 + \{3\varepsilon\sin\beta_1 - \varepsilon_2\sin(\alpha_2-\beta_1)\}\,\Phi_{2,1} - \varepsilon_1\sin(\beta_2-\alpha_1)\,\Phi_{2,2}\right], \tag{223}$$

$$W_{4,2} = -\frac{1}{4na^2}\left[3\varepsilon_2\cos\alpha_2\Phi_2 + \varepsilon_1\cos(\alpha_1+\beta_1)\,\Phi_{2,1} + 3\varepsilon\cos\beta_2\Phi_{2,2}\right], \tag{224}$$

$$W'_{4,2} = -\frac{1}{4na^2}\left[3\varepsilon_2\sin\alpha_2\Phi_2 + \varepsilon_1\sin(\alpha_1+\beta_1)\,\Phi_{2,1} + 3\varepsilon\sin\beta_2\Phi_{2,2}\right], \tag{225}$$

$$W_{4,3} = -\frac{1}{4na^2}\left[\varepsilon_2\cos(\alpha_2+\beta_1)\,\Phi_{2,1} + \varepsilon_1\cos(\beta_2+\alpha_1)\,\Phi_{2,2}\right], \tag{226}$$

$$W'_{4,3} = -\frac{1}{4na^2}\left[\varepsilon_2\sin(\alpha_2+\beta_1)\,\Phi_{2,1} + \varepsilon_1\sin(\beta_2+\alpha_1)\,\Phi_{2,2}\right], \tag{227}$$

$$W_{4,4} = -\frac{1}{4na^2}\,\varepsilon_2\cos(\alpha_2+\beta_2)\,\Phi_{2,2}, \tag{228}$$

$$W'_{4,4} = -\frac{1}{4na^2}\,\varepsilon_2\sin(\alpha_2+\beta_2)\,\Phi_{2,2}. \tag{229}$$

§50. If the given system of applied forces acted on a perfect sphere we should have two classes of displacements, viz. what I have elsewhere called "mixed radial and transverse" and "pure transverse." The former would depend on the V, R and Θ forces, the latter on the Φ forces.

The principal terms in the present case are of course of these respective forms. As regards the subsidiary terms, Y_0 in (187) answers to "pure radial" displacements, and depends only on V, R and Θ forces. The constants determined by equations (188)–(192) are merely the increments required to give the second approximations to those principal terms which depend on the V, R and Θ forces, and their values are independent of the Φ forces.

The constants given by (200)–(208) are likewise independent of the Φ forces and answer to displacements of the type "mixed radial and transverse." Thus the constants of the Y and Z types whose first suffix is even answer to applied forces of the "mixed radial and transverse" type.

The Y and Z constants, however, in which the first suffix is 3, are shown by equations (193) to (199) to depend solely on the Φ forces; thus the corresponding terms, though of the type "mixed radial and transverse," are due to purely transverse surface forces.

The constants determined by (209) to (213) are the increments required to give second approximations to the principal terms depending on the Φ forces, and their values depend solely on the Φ forces. The constants given by (221) to (229) likewise depend solely on the Φ forces and answer to pure transverse displacements. Thus the constants of the W type whose first suffix is even, answer to applied forces of the pure transverse type.

The W constants, however, in which the first suffix is 3, are shown by equations (214) to (220) to depend on the V, R, Θ forces; thus the corresponding displacements, though of the type pure transverse, are due to "mixed radial and transverse" forces. The species of reciprocity thus found to exist between the radial and transverse and the pure transverse forces seems rather striking.

§51. The equations determining the Y and Z constants whose first suffix is even may appear long and complicated. A little examination shows, however, that they are of marvellous simplicity. Take for instance (188). The quantities $\overline{L}$, $\overline{M}$, $\overline{N}$, $\overline{O}$ are precisely the same functions of V_2, R_2, Θ_2 that $\overline{L_1}$, $\overline{M_1}$, $\overline{N_1}$, $\overline{O_1}$ are of $V_{2,1}$, $R_{2,1}$, $\Theta_{2,1}$, and that $\overline{L_2}$, $\overline{M_2}$, $\overline{N_2}$, $\overline{O_2}$ are of $V_{2,2}$, $R_{2,2}$, $\Theta_{2,2}$. Thus if we determine the terms in ϵ in δY_2 and δZ_2 from the equations

$$\left.\begin{aligned} A_2\delta Y_2 + B_2\delta Z_2 &= -\frac{2}{7}\,\epsilon\left(a\frac{d\overline{L}}{da} - 3\overline{M}\right), \\ C_2\delta Y_2 + D_2\delta Z_2 &= -\frac{1}{7}\,\epsilon\left(a\frac{d\overline{M}}{da} - \overline{N} - 3\overline{O}\right), \end{aligned}\right\} \qquad (230)$$

we get the terms in ϵ_1 by simply substituting $\frac{1}{42}\,\epsilon_1\cos(\alpha_1 - \beta_1)$ for $\frac{1}{7}\,\epsilon$, and replacing V_2, R_2, Θ_2 by $V_{2,1}$, $R_{2,1}$, $\Theta_{2,1}$. For the terms in ϵ_2 we write $-\frac{4}{21}\,\epsilon_2\cos(\alpha_2 - \beta_2)$ for $\frac{1}{7}\,\epsilon$, and replace V_2, R_2, Θ_2 by $V_{2,2}$, $R_{2,2}$, $\Theta_{2,2}$.

Nor is this all. We see that in (188) to (192), i. e. all the equations in which the surface harmonic is of the second degree, factors $a\frac{d\overline{L}_s}{da}-3\overline{M}_s$, where $s=0,\ 1,\ 2$, occur in the first equations, and factors $a\frac{d\overline{M}_s}{da}-\overline{N}_s-3\overline{O}_s$ in the second; and that the numerical coefficient of each ε in the first of each pair of equations is double that of the corresponding ε in the second. Thus by mere inspection of the values of δY_2 and δZ_2 given by (230) we can write down the complete value of every Y or Z constant in which the first suffix is 2.

Corresponding results hold for the surface harmonics of the third and fourth degrees. Thus the complete solutions of all the equations (200) to (208) may be written down by simple inspection of the values of Y_4 and Z_4 supplied by

$$A_4Y_4+B_4Z_4=-\frac{18}{35}\varepsilon\left(a\frac{d\overline{L}}{da}+4\overline{M}\right),$$

$$C_4Y_4+D_4Z_4=-\frac{9}{35}\varepsilon\left(a\frac{d\overline{M}}{da}-\overline{N}+4\overline{O}\right).$$

§52. The first preliminary to the determination of the Y and Z constants in the subsidiary terms is to obtain the first approximations to the values of Y_2, Z_2, $Y_{2,1}$, $Z_{2,1}$, $Y_{2,2}$, $Z_{2,2}$ as given by (84), (85), (86), (114), (115) and (116).

From (84) and (114) we get

$$\left.\begin{aligned} Y_2&=\frac{1}{19m-5n}\left[21a^{-2}(R_2-2\Theta_2)+\frac{3(7m-n)}{m+n}\rho V_2\right],\\ Z_2&=\frac{1}{n(19m-5n)}\left[2(4m-n)a^2\rho V_2+3(m-n)\Theta_2+(8m-n)R_2\right].\end{aligned}\right\}\quad(231)$$

From these we deduce the values of $Y_{2,1}$ and $Z_{2,1}$ by simply writing $V_{2,1}$, $R_{2,1}$, $\Theta_{2,1}$ for V_2, R_2, Θ_2, and the values of $Y_{2,2}$ and $Z_{2,2}$ by writing $V_{2,2}$, etc., for V_2, etc.

We next substitute these results in coefficients such as $a\frac{d\overline{L}}{da}+4\overline{M}$ in (200). The values of $\overline{L}$, etc., are given in §5, and it will be remembered that we treat Y and Z coefficients as constants in differentiations such as $\frac{d\overline{L}}{da}$.

It will suffice to record the values of $\overline{L}$, $\overline{M}$, etc., as the values of $\overline{L}_1$, $\overline{M}_1$, etc., and of $\overline{L}_2$, $\overline{M}_2$, etc., may be deduced by writing $V_{2,1}$, etc., for V_2, etc., in the former case and $V_{2,2}$, etc., for V_2, etc., in the latter. We have

$$\overline{L} = R_2, \tag{232}$$

$$\overline{M} = \Theta_2, \tag{233}$$

$$\overline{N} = \frac{1}{19m - 5n}\{4(3m - n)\,a^2\rho V_2 + (31m - 17n)R_2 - 24(m - n)\Theta_2\}, \tag{234}$$

$$\overline{O} = \frac{1}{19m - 5n}\{(3m - n)\,a^2\rho V_2 + 3(m - n)R_2 + (13m + n)\Theta_2\}, \tag{235}$$

$$a\frac{d\overline{L}}{da} = \frac{1}{19m - 5n}\{-8(4m - n)\,a^2\rho V_2 + 6(m - n)(R_2 - 2\Theta_2)\}, \tag{236}$$

$$a\frac{d\overline{M}}{da} = \frac{1}{19m - 5n}\{-4(4m - n)\,a^2\rho V_2 - 2(8m - n)(R_2 - 2\Theta_2)\}, \tag{237}$$

$$a\frac{d\overline{L}}{da} - 6\overline{M} = \frac{1}{19m - 5n}\{-8(4m - n)\,a^2\rho V_2 + 6(m - n)R_2 - 42(3m - n)\Theta_2\}, \tag{238}$$

$$a\frac{d\overline{L}}{da} - 3\overline{M} = \frac{1}{19m - 5n}\{-8(4m - n)\,a^2\rho V_2 + 6(m - n)R_2 - 3(23m - 9n)\Theta_2\}, \tag{239}$$

$$a\frac{d\overline{M}}{da} - \overline{N} - 3\overline{O} = \frac{1}{19m - 5n}\{-(37m - 11n)\,a^2\rho V_2 - 28(2m - n)R_2 + (17m - 31n)\Theta_2\}, \tag{240}$$

$$a\frac{d\overline{L}}{da} + 4\overline{M} = \frac{1}{19m - 5n}\{-8(4m - n)\,a^2\rho V_2 + 6(m - n)R_2 + 8(8m - n)\Theta_2\}, \tag{241}$$

$$a\frac{d\overline{M}}{da} - \overline{N} + 4\overline{O} = \frac{1}{19m - 5n}\{-4(4m - n)\,a^2\rho V_2 - 7(5m - n)R_2 + 12(9m - 2n)\Theta_2\}, \tag{242}$$

$$a\frac{d\overline{M}}{da} + \overline{N} = \frac{1}{19m - 5n}\{-4ma^2\rho V_2 + 15(m - n)R_2 + 4(2m + 5n)\Theta_2\}, \tag{243}$$

$$a\frac{d\overline{M}}{da} + \overline{N} - 5\overline{O} = -(a^2\rho V_2 + 3\Theta_2). \tag{244}$$

§53. Employing (238) in (187) we have

$$Y_0 = 3\epsilon\varkappa_2 + \epsilon_1 \cos(\alpha_1 - \beta_1)\,\varkappa_{2,1} + 4\epsilon_2 \cos(\alpha_2 - \beta_2)\,\varkappa_{2,2}, \tag{245}$$

where

$$\varkappa_2 = \frac{2\{4(4m - n)\,a^2\rho V_2 - 3(m - n)R_2 + 21(3m - n)\Theta_2\}}{5(3m - n)(19m - 5n)}, \tag{246}$$

while $\varkappa_{2,1}$ and $\varkappa_{2,2}$ are derived from $\varkappa_2$ by writing 2,1 and 2,2 respectively for 2 in the suffixes of V, R, Θ.

With a similar suffix notation, let

$$\lambda_2 = \frac{a^{-2}}{(19m-5n)^2}\{-(5m-3n)a^2\rho V_2 - 2(31m-17n)R_2 + 2(43m-29n)\Theta_2\}, \tag{247}$$

$$\lambda_2' = \frac{7(89m^2-48mn+7n^2)a^2\rho V_2 + 72(m-n)^2R_2 + 3(13m+n)(27m-13n)\Theta_2}{42n(19m-5n)^2}, \tag{248}$$

then we get

$$\left.\begin{aligned} \delta Y_2 &= 6\epsilon\lambda_2 + \epsilon_1\cos(\alpha_1-\beta_1)\lambda_{2,1} - 8\epsilon_2\cos(\alpha_2-\beta_2)\lambda_{2,2}, \\ \delta Z_2 &= 6\epsilon\lambda_2' + \epsilon_1\cos(\alpha_1-\beta_1)\lambda_{2,1}' - 8\epsilon_2\cos(\alpha_2-\beta_2)\lambda_{2,2}', \end{aligned}\right\} \tag{249}$$

$$\left.\begin{aligned} \delta Y_{2,1} &= 3[\epsilon_1\cos\alpha_1\lambda_2 + \{\epsilon\cos\beta_1 + 2\epsilon_2\cos(\alpha_2-\beta_1)\}\lambda_{2,1} + 2\epsilon_1\cos(\beta_2-\alpha_1)\lambda_{2,2}], \\ \delta Z_{2,1} &= 3[\epsilon_1\cos\alpha_1\lambda_2' + \{\epsilon\cos\beta_1 + 2\epsilon_2\cos(\alpha_2-\beta_1)\}\lambda_{2,1}' + 2\epsilon_1\cos(\beta_2-\alpha_1)\lambda_{2,2}'], \end{aligned}\right\} \tag{250}$$

$$\left.\begin{aligned} \delta Y_{2,1}' &= 3[\epsilon_1\sin\alpha_1\lambda_2 + \{\epsilon\sin\beta_1 + 2\epsilon_2\sin(\alpha_2-\beta_1)\}\lambda_{2,1} + 2\epsilon_1\sin(\beta_2-\alpha_1)\lambda_{2,2}], \\ \delta Z_{2,1}' &= 3[\epsilon_1\sin\alpha_1\lambda_2' + \{\epsilon\sin\beta_1 + 2\epsilon_2\sin(\alpha_2-\beta_1)\}\lambda_{2,1}' + 2\epsilon_1\sin(\beta_2-\alpha_1)\lambda_{2,2}'], \end{aligned}\right\} \tag{251}$$

$$\left.\begin{aligned} \delta Y_{2,2} &= \frac{3}{2}[-4\epsilon_2\cos\alpha_2\lambda_2 + \epsilon_1\cos(\alpha_1+\beta_1)\lambda_{2,1} - 4\epsilon\cos\beta_2\lambda_{2,2}], \\ \delta Z_{2,2} &= \frac{3}{2}[-4\epsilon_2\cos\alpha_2\lambda_2' + \epsilon_1\cos(\alpha_1+\beta_1)\lambda_{2,1}' - 4\epsilon\cos\beta_2\lambda_{2,2}'], \end{aligned}\right\} \tag{252}$$

$$\left.\begin{aligned} \delta Y_{2,2}' &= \frac{3}{2}[-4\epsilon_2\sin\alpha_2\lambda_2 + \epsilon_1\sin(\alpha_1+\beta_1)\lambda_{2,1} - 4\epsilon\sin\beta_2\lambda_{2,2}], \\ \delta Z_{2,2}' &= \frac{3}{2}[-4\epsilon_2\sin\alpha_2\lambda_2' + \epsilon_1\sin(\alpha_1+\beta_1)\lambda_{2,1}' - 4\epsilon\sin\beta_2\lambda_{2,2}']. \end{aligned}\right\} \tag{253}$$

For the Y and Z constants whose first suffix is 3 we easily find from equations (193) to (199),

$$\begin{aligned} Y_3 &= -\frac{54a^{-2}n}{12m-5n}Z_3 \\ &= \frac{108a^{-3}}{5(33m-7n)}\{\epsilon_1\sin(\alpha_1-\beta_1)\Phi_{2,1} - 2\epsilon_2\sin(\alpha_2-\beta_2)\Phi_{2,2}\}, \end{aligned} \tag{254}$$

$$\begin{aligned} Y_{3,1} = -\frac{54a^{-2}n}{12m-5n}Z_{3,1} = \frac{81a^{-3}}{33m-7n}[&-\epsilon_1\sin\alpha_1\Phi_2 \\ &+ \{\epsilon\sin\beta_1 + \epsilon_2\sin(\alpha_2-\beta_1)\}\Phi_{2,1} - \epsilon_1\sin(\beta_2-\alpha_1)\Phi_{2,2}], \end{aligned} \tag{255}$$

$$\begin{aligned} Y_{3,1}' = -\frac{54a^{-2}n}{12m-5n}Z_{3,1}' = \frac{81a^{-3}}{33m-7n}[&\epsilon_1\cos\alpha_1\Phi_2 - \{\epsilon\cos\beta_1 + \epsilon_2\cos(\alpha_2-\beta_1)\}\Phi_{2,1} \\ &+ \epsilon_1\cos(\beta_2-\alpha_1)\Phi_{2,2}], \end{aligned} \tag{256}$$

$$Y_{3,2} = -\frac{54a^{-2}n}{12m-5n}Z_{3,2} = \frac{162a^{-3}}{33m-7n}[-\epsilon_2\sin\alpha_2\Phi_2 + \epsilon\sin\beta_2\Phi_{2,2}], \tag{257}$$

$$Y'_{3,2} = -\frac{54a^{-2}n}{12m-5n} Z'_{3,2} = \frac{162a^{-3}}{33m-7n}\left[\varepsilon_2\cos\alpha_2\Phi_2 - \varepsilon\cos\beta_2\Phi_{2,2}\right], \tag{258}$$

$$\begin{aligned} Y_{3,3} = -\frac{54a^{-2}n}{12m-5n} Z_{3,3} = \frac{27a^{-3}}{33m-7n}\big[&-\varepsilon_2\sin(\alpha_2+\beta_1)\Phi_{2,1} \\ &+\varepsilon_1\sin(\beta_2+\alpha_1)\Phi_{2,2}\big], \end{aligned} \tag{259}$$

$$\begin{aligned} Y'_{3,3} = -\frac{54a^{-2}n}{12m-5n} Z'_{3,3} = \frac{27a^{-3}}{33m-7n}\big[&\varepsilon_2\cos(\alpha_2+\beta_1)\Phi_{2,1} \\ &-\varepsilon_1\cos(\beta_2+\alpha_1)\Phi_{2,2}\big]. \end{aligned} \tag{260}$$

For the Y and Z constants whose first suffix is 4, let

$$\mu_2 = \frac{22a^{-4}(2\Theta_2 - R_2)}{21(17m-3n)}, \tag{261}$$

$$\begin{aligned} \mu'_2 = \{&12(4m-n)(17m-3n)\rho V_2 - (1863m^2 - 440mn - 23n^2)a^{-2}R_2 \\ &+ 4(447m^2 - 7mn - 34n^2)a^{-2}\Theta_2\} \div \{630n(17m-3n)(19m-5n)\}, \end{aligned} \tag{262}$$

etc., then

$$\left.\begin{aligned} Y_4 &= 36\varepsilon\mu_2 - 8\varepsilon_1\cos(\alpha_1-\beta_1)\mu_{2,1} + 8\varepsilon_2\cos(\alpha_2-\beta_2)\mu_{2,2}, \\ Z_4 &= 36\varepsilon\mu'_2 - 8\varepsilon_1\cos(\alpha_1-\beta_1)\mu'_{2,1} + 8\varepsilon_2\cos(\alpha_2-\beta_2)\mu'_{2,2}, \end{aligned}\right\} \tag{263}$$

$$\left.\begin{aligned} Y_{4,1} = 105\varepsilon_1\cos\alpha_1\mu_2 + 35\{3\varepsilon\cos\beta_1 - \varepsilon_2\cos(\alpha_2-\beta_1)\}\,&\mu_{2,1} \\ - 35\varepsilon_1\cos(\beta_2-\alpha_1)\,&\mu_{2,2}, \\ Z_{4,1} = 105\varepsilon_1\cos\alpha_1\mu'_2 + 35\{3\varepsilon\cos\beta_1 - \varepsilon_2\cos(\alpha_2-\beta_1)\}\,&\mu'_{2,1} \\ - 35\varepsilon_1\cos(\beta_2-\alpha_1)\,&\mu'_{2,2}, \end{aligned}\right\} \tag{264}$$

$$\left.\begin{aligned} Y'_{4,1} = 105\varepsilon_1\sin\alpha_1\mu_2 + 35\{3\varepsilon\sin\beta_1 - \varepsilon_2\sin(\alpha_2-\beta_1)\}\,&\mu_{2,1} \\ - 35\varepsilon_1\sin(\beta_2-\alpha_1)\,&\mu_{2,2}, \\ Z'_{4,1} = 105\varepsilon_1\sin\alpha_1\mu'_2 + 35\{3\varepsilon\sin\beta_1 - \varepsilon_2\sin(\alpha_2-\beta_1)\}\,&\mu'_{2,1} \\ - 35\varepsilon_1\sin(\beta_2-\alpha_1)\,&\mu'_{2,2}, \end{aligned}\right\} \tag{265}$$

$$\left.\begin{aligned} Y_{4,2} &= 35\left[3\varepsilon_2\cos\alpha_2\mu_2 + \varepsilon_1\cos(\alpha_1+\beta_1)\mu_{2,1} + 3\varepsilon\cos\beta_2\mu_{2,2}\right], \\ Z_{4,2} &= 35\left[3\varepsilon_2\cos\alpha_2\mu'_2 + \varepsilon_1\cos(\alpha_1+\beta_1)\mu'_{2,1} + 3\varepsilon\cos\beta_2\mu'_{2,2}\right], \end{aligned}\right\} \tag{266}$$

$$\left.\begin{aligned} Y'_{4,2} &= 35\left[3\varepsilon_2\sin\alpha_2\mu_2 + \varepsilon_1\sin(\alpha_1+\beta_1)\mu_{2,1} + 3\varepsilon\sin\beta_2\mu_{2,2}\right], \\ Z'_{4,2} &= 35\left[3\varepsilon_2\sin\alpha_2\mu'_2 + \varepsilon_1\sin(\alpha_1+\beta_1)\mu'_{2,1} + 3\varepsilon\sin\beta_2\mu'_{2,2}\right], \end{aligned}\right\} \tag{267}$$

$$\left.\begin{aligned} Y_{4,3} &= 35\left[\varepsilon_2\cos(\alpha_2+\beta_1)\mu_{2,1} + \varepsilon_1\cos(\beta_2+\alpha_1)\mu_{2,2}\right], \\ Z_{4,3} &= 35\left[\varepsilon_2\cos(\alpha_2+\beta_1)\mu'_{2,1} + \varepsilon_1\cos(\beta_2+\alpha_1)\mu'_{2,2}\right], \end{aligned}\right\} \tag{268}$$

$$\left.\begin{aligned} Y'_{4,3} &= 35\left[\varepsilon_2\sin(\alpha_2+\beta_1)\mu_{2,1} + \varepsilon_1\sin(\beta_2+\alpha_1)\mu_{2,2}\right], \\ Z'_{4,3} &= 35\left[\varepsilon_2\sin(\alpha_2+\beta_1)\mu'_{2,1} + \varepsilon_1\sin(\beta_2+\alpha_1)\mu'_{2,2}\right], \end{aligned}\right\} \tag{269}$$

$$\left.\begin{aligned} Y_{4,4}/\cos(\alpha_2+\beta_2) &= Y'_{4,4}/\sin(\alpha_2+\beta_2) = 35\varepsilon_2\mu_{2,2}, \\ Z_{4,4}/\cos(\alpha_2+\beta_2) &= Z'_{4,4}/\sin(\alpha_2+\beta_2) = 35\varepsilon_2\mu'_{2,2}. \end{aligned}\right\} \tag{270}$$

In equations (209) to (213) and (221) to (229) will be found the values of all the subsidiary W constants except those whose first suffix is 3. For these, having regard to (243), and putting

$$\nu_2 = \frac{-4ma^2\rho V_2 + 15(m-n)R_2 + 4(2m+5n)\Theta_2}{120a^2n(19m-5n)}, \tag{271}$$

etc., we have

$$W_3 = 4\{\epsilon_1 \sin(\alpha_1 - \beta_1)\nu_2 - 2\epsilon_2 \sin(\alpha_2 - \beta_2)\nu_{2,2}\}, \tag{272}$$

$$W_{3,1} = 15[-\epsilon_1 \sin\alpha_1\nu_2 + \{\epsilon\sin\beta_1 + \epsilon_2\sin(\alpha_2 - \beta_1)\}\nu_{2,1} - \epsilon_1\sin(\beta_2 - \alpha_1)\nu_{2,2}], \tag{273}$$

$$W'_{3,1} = 15[\epsilon_1\cos\alpha_1\nu_2 - \{\epsilon\cos\beta_1 + \epsilon_2\cos(\alpha_2 - \beta_1)\}\nu_{2,1} + \epsilon_1\cos(\beta_2 - \alpha_1)\nu_{2.2}], \tag{274}$$

$$W_{3,2} = 30[-\epsilon_2\sin\alpha_2\nu_2 + \epsilon\sin\beta_2\nu_{2,2}], \tag{275}$$

$$W'_{3,2} = 30[\epsilon_2\cos\alpha_2\nu_2 - \epsilon\cos\beta_2\nu_{2,2}], \tag{276}$$

$$W_{3,3} = 5[-\epsilon_2\sin(\alpha_2 + \beta_1)\nu_{2,1} + \epsilon_1\sin(\beta_2 + \alpha_1)\nu_{2,2}], \tag{277}$$

$$W'_{3,3} = 5[\epsilon_2\cos(\alpha_2 + \beta_1)\nu_{2,1} - \epsilon_1\cos(\beta_2 + \alpha_1)\nu_{2,2}]. \tag{278}$$

§54. All the subsidiary constants have thus had their values determined and recorded. The expressions for the displacements in the general case would occupy rather much space. In any specified case, however, any one will be able to write down the values of the displacements at once by reference to the values of the arbitrary constants given above. If the notation be not fresh in his memory he need only refer to the general solution as given by (1), (2), (3) and (4) of Section I, to the definition of $P_{i,k}$ in (1) of Section III, and to the explanation of the dashed and undashed coefficients in §35.

§55. If the harmonics in the expressions for the applied forces are identical in phase with those in the surface equation, a simplification occurs. For putting

$$\left.\begin{array}{l} X'_{2,1} = X_{2,1} \text{ or } \beta_1 = \alpha_1, \\ X'_{2,2} = X_{2,2} \text{ or } \beta_2 = \alpha_2, \end{array}\right\} \tag{279}$$

we get $$Y_3 = Z_3 = W_3 = 0,$$

and the formulæ for the other constants whose first suffix is 3 are simplified.

If in addition

$$\Phi_2/\epsilon = \Phi_{2,1}/\epsilon_1 = \Phi_{2,2}/\epsilon_2, \tag{280}$$

then all the Y and Z constants whose first suffix is 3 vanish; in this case the pure transverse surface forces give rise to no "mixed radial and transverse" subsidiary terms.

If on the other hand while

$$\beta_1 = \alpha_1 \text{ and } \beta_2 = \alpha_2,$$

we have

$$\left.\begin{aligned} V_2/\varepsilon &= V_{2,1}/\varepsilon_1 = V_{2,2}/\varepsilon_2, \\ R_2/\varepsilon &= R_{2,1}/\varepsilon_1 = R_{2,2}/\varepsilon_2, \\ \Theta_2/\varepsilon &= \Theta_{2,1}/\varepsilon_1 = \Theta_{2,2}/\varepsilon_2, \end{aligned}\right\} \tag{281}$$

then all the W constants whose first suffix is 3 vanish; in this case the "mixed radial and transverse" applied forces give rise to no pure transverse subsidiary terms.

Since μ_2 is by (261) independent of V_2, it follows that if bodily forces alone exist, all the Y constants whose first suffix is 4 vanish. The same result follows if

$$R_2 = 2\Theta_2,$$

or the applied surface forces come from a potential $\Theta_2 r^2/a$.

From (245) we see that the pure radial displacement vanishes if ε be zero while

$$\alpha_1 \sim \beta_1 = \alpha_2 \sim \beta_2 = \pi/2.$$

§56. Before it can be maintained that we have reached a complete solution of the problem proposed in §26, the promise to give a good account of the arbitrary constants appearing in the "uncompensated" terms must be redeemed.

With a view to this, let

$$X = \frac{8}{15}\pi a^2 \left[-3\varepsilon_1 \sin\alpha_1 \Phi_2 + \{3\varepsilon \sin\beta_1 - 2\varepsilon_2 \sin(\alpha_2 - \beta_1)\}\Phi_{2,1} \right.$$
$$\left. + 2\varepsilon_1 \sin(\beta_2 - \alpha_1)\Phi_{2,2}\right], \tag{282}$$

$$Y = \frac{8}{15}\pi a^2 \left[3\varepsilon_1 \cos\alpha_1 \Phi_2 - \{3\varepsilon \cos\beta_1 - 2\varepsilon_2 \cos(\alpha_2 - \beta_1)\}\Phi_{2,1} \right.$$
$$\left. - 2\varepsilon_1 \cos(\beta_2 - \alpha_1)\Phi_{2,2}\right], \tag{283}$$

$$Z = \frac{8}{15}\pi a^2 \left[\varepsilon_1 \sin(\alpha_1 - \beta_1)\Phi_{2,1} + 8\varepsilon_2 \sin(\alpha_2 - \beta_2)\Phi_{2,2}\right], \tag{284}$$

$$L = \frac{4\pi a^3}{15}\left[3\varepsilon_1 \sin\alpha_1 (a^2\rho V_2 + 3\Theta_2) - \{3\varepsilon \sin\beta_1 - 2\varepsilon_2 \sin(\alpha_2 - \beta_1)\}(a^2\rho V_{2,1} + 3\Theta_{2,1}) \right.$$
$$\left. - 2\varepsilon_1 \sin(\beta_2 - \alpha_1)(a^2\rho V_{2,2} + 3\Theta_{2,2})\right], \tag{285}$$

$$M = \frac{4\pi a^3}{15}\left[-3\varepsilon_1 \cos\alpha_1 (a^2\rho V_2 + 3\Theta_2) + \{3\varepsilon \cos\beta_1 - 2\varepsilon_2 \cos(\alpha_2 - \beta_1)\}(a^2\rho V_{2,1} + 3\Theta_{2,1}) \right.$$
$$\left. + 2\varepsilon_1 \cos(\beta_2 - \alpha_1)(a^2\rho V_{2,2} + 3\Theta_{2,2})\right], \tag{286}$$

$$N = -\frac{4\pi a^3}{15}\left[\varepsilon_1 \sin(\alpha_1 - \beta_1)(a^2\rho V_{2,1} + 3\Theta_{2,1}) \right.$$
$$\left. + 8\varepsilon_2 \sin(\alpha_2 - \beta_2)(a^2\rho V_{2,2} + 3\Theta_{2,2})\right]. \tag{287}$$

Then employing (244), we see that the results (186) are equivalent to the following:

$$\frac{\mathrm{a}_1}{X}=\frac{\mathrm{b}_1}{Y}=\frac{\mathrm{c}_1}{Z}=\frac{\mathrm{a}_2}{-\frac{3}{2}X}=\frac{\mathrm{b}_2}{-\frac{3}{2}Y}=\frac{\mathrm{c}_2}{\frac{3}{2}Z}=\frac{\mathrm{a}_3}{\frac{3}{2}X}=\frac{\mathrm{b}_3}{-\frac{3}{2}Y}=\frac{3}{8\pi a^2}, \tag{288}$$

$$\frac{\mathrm{d}_2}{-M}=\frac{\mathrm{e}_2}{L}=\frac{\mathrm{d}_3}{M}=\frac{\mathrm{e}_3}{L}=\frac{\mathrm{f}_3}{-N}=\frac{3}{8\pi a^3}. \tag{289}$$

Thus the terms remaining uncompensated are:

on the left of the first surface condition,

$$R=\frac{3}{8\pi a^2}\left[X\sin\theta\cos\phi+Y\sin\theta\sin\phi+Z\cos\theta\right]; \tag{290}$$

on the left of the second surface condition,

$$\Theta=\frac{3}{8\pi a^2}\left[-\frac{3}{2}(X\cos\theta\cos\phi+Y\cos\theta\sin\phi-Z\sin\theta)+\frac{L}{a}\sin\phi-\frac{M}{a}\cos\phi\right]; \tag{291}$$

on the left of the third surface condition,

$$\Phi=\frac{3}{8\pi a^2}\left[\frac{3}{2}(X\sin\phi-Y\cos\phi)\right.$$
$$\left.+\frac{L}{a}\cos\theta\cos\phi+\frac{M}{a}\cos\theta\sin\phi-\frac{N}{a}\sin\theta\right]. \tag{292}$$

It is a simple matter to show that if forces act over a spherical surface of radius a, the components per unit of surface along the directions r, θ, ϕ having the magnitudes R, Θ, Φ given by (290), (291) and (292) respectively, then the resultant consists of a force whose components along the axes of x, y, z at the centre of the sphere are $-X$, $-Y$, $-Z$, and of a couple whose components about these axes are $-L$, $-M$, $-N$.

§57. The bearing of the results (288) and (289) will now be appreciated at once when we state that the statical resultant of the applied forces as given by (4) and (5) consists of the force whose components along the axes of x, y, z at the centre are $+X$, $+Y$ and $+Z$, and of the couple whose components about these axes are $+L$, $+M$ and $+N$.

It will suffice to illustrate the proof of this statement by finding the component couple about the axis of x.

Taking first the bodily forces (4), they give at any point the following components per unit mass:

$$\begin{aligned} \text{parallel to } x &= -xV_2 + z\cos\beta_1 V_{2,1} + 2(x\cos\beta_2 + y\sin\beta_2)V_{2,2}, \\ \text{`` \quad ``} \quad y &= -yV_2 + z\sin\beta_1 V_{2,1} + 2(x\sin\beta_2 - y\cos\beta_2)V_{2,2}, \\ \text{`` \quad ``} \quad z &= 2zV_2 + (x\cos\beta_1 + y\sin\beta_1)V_{2,1}. \end{aligned}$$

The corresponding elementary couple about the axis of x is

$$3yzV_2 + \{xy\cos\beta_1 + (y^2 - z^2)\sin\beta_1\}V_{2,1} + 2z(y\cos\beta_2 - x\sin\beta_2)V_{2,2}.$$

Thus the resultant couple about $ox =$

$$\begin{aligned} \int\!\!\int\!\!\int d\phi\, d\theta\, dr\, \rho \sin\theta r^4 &\left[3\sin\theta\cos\theta\sin\phi V_2\right. \\ &\left.+\{\sin^2\theta\sin\phi\cos(\phi-\beta_1) - \cos^2\theta\sin\beta_1\}V_{2,1} + 2\sin\theta\cos\theta\sin(\phi-\beta_2)V_{2,2}\right], \end{aligned}$$

the integration extending throughout the entire mass.

Now at the surface

$$r = a\{1 + \epsilon P_2 + \epsilon_1 P_{2,1}\cos(\phi - \alpha_1) + \epsilon_2 P_{2,2}\cos(2\phi - \alpha_2)\};$$

thus neglecting squares and products of ϵ, ϵ_1, ϵ_2, we get couple about $ox =$

$$\begin{aligned} \frac{\rho a^5}{5}\int_0^\pi\!\!\int_0^{2\pi} d\theta\, d\phi \sin\theta &\left[1 + 5\left\{\epsilon\frac{3\cos^2\theta - 1}{2} + \epsilon_1\sin\theta\cos\theta\cos(\phi - \alpha_1)\right.\right. \\ &\left.\left.+\epsilon_2\sin^2\theta\cos(2\phi - \alpha_2)\right\}\right]\left[3\sin\theta\cos\theta\sin\phi V_2\right. \\ &\left.+\{\sin^2\theta\sin\phi\cos(\phi-\beta_1) - \cos^2\theta\sin\beta_1\}V_{2,1} + 2\sin\theta\cos\theta\sin(\phi-\beta_2)V_{2,2}\right]. \end{aligned}$$

The terms independent of ϵ, ϵ_1 or ϵ_2 of course vanish, and any other term containing an odd power of a sine or cosine of ϕ, or an odd power of $\cos\theta$, may obviously be neglected. The terms which do not vanish reduce very simply to

$$\frac{4}{15}\pi\rho a^5\left[3\epsilon_1\sin\alpha_1 V_2 - \{3\epsilon\sin\beta_1 - 2\epsilon_2\sin(\alpha_2 - \beta_1)\}V_{2,1} - 2\epsilon_1\sin(\beta_2 - \alpha_1)V_{2,2}\right].$$

Taking next the surface forces with Θ coefficients, i. e. forces

$$\frac{d}{d\theta}\{\Theta_2 P_2 + \Theta_{2,1}\cos(\phi - \beta_1)P_{2,1} + \Theta_{2,2}\cos(2\phi - \beta_2)P_{2,2}\} \text{ along } \theta,$$

$$\frac{1}{\sin\theta}\frac{d}{d\phi}\{\Theta_{2,1}\cos(\phi - \beta_1)P_{2,1} + \Theta_{2,2}\cos(2\phi - \beta_2)P_{2,2}\} \text{ along } \phi,$$

we find for the total couple about the axis of x

$$a^3\int^{\pi}\int_0^{2\pi} d\theta\, d\phi \sin\theta \left[1 + 3\left\{\varepsilon\frac{3\cos^2\theta - 1}{2} + \varepsilon_1 \sin\theta\cos\theta\cos(\phi - \alpha_1)\right.\right.$$
$$\left.\left. + \varepsilon_2 \sin^2\theta\cos(2\phi - \alpha_2)\right\}\right]\left[3\sin\theta\cos\theta\sin\phi\Theta_2\right.$$
$$\left. + \frac{1}{2}\{\sin^2\theta\sin(2\phi - \beta_1) + \sin\beta_1(3\sin^2\theta - 2)\}\Theta_{2,1} + 2\sin\theta\cos\theta\sin(\phi - \beta_2)\Theta_{2,2}\right].$$

This easily reduces to

$$\frac{4}{5}\pi a^3\left[3\varepsilon_1\sin\alpha_1\Theta_2 - \{3\varepsilon\sin\beta_1 - 2\varepsilon_2\sin(\alpha_2 - \beta_1)\}\Theta_{2,1} - 2\varepsilon_1\sin(\beta_2 - \alpha_1)\Theta_{2,2}\right].$$

The radial forces all have their lines of action passing through the centre and so obviously give no couple. It is also easily found that no couple arises from the surface forces with Φ coefficients. Thus the complete couple about the axis of x is got by adding the couples arising from the V and the Θ forces, and referring to (285) we see at once that this sum amounts simply to $+L$.

The proofs that $+M$ and N are the component couples about the two other axes, and that $+X$, $+Y$ and $+Z$ are the components of the resultant force of the applied system, proceed on exactly similar lines.

If the applied system of forces is in equilibrium, X, Y, Z, L, M, N, as is well known, must all vanish; and consequently by (288) and (289) every single "uncompensated" term vanishes. Thus in any case of equilibrium there are no "uncompensated" terms left over and the completeness of the solution is thus firmly established.

§58. The previous method, it may be mentioned, can be applied to the shell whose two surfaces are

$$r = a\{1 + \varepsilon P_2 + \varepsilon_1 P_{2,1}\cos(\phi - \alpha_1) + \varepsilon_2 P_{2,2}\cos(2\phi - \alpha_2)\},$$
$$r = b\{1 + \varepsilon' P_2 + \varepsilon_1' P_{2,1}\cos(\phi - \alpha_1') + \varepsilon_2' P_{2,2}\cos(2\phi - \alpha_2')\},$$

acted on by bodily forces derivable from a potential

$$r^2\{V_2P_2 + V_{2,1}P_{2,1}\cos(\phi - \beta_1) + V_{2,2}P_{2,2}\cos(2\phi - \beta_2)\}$$
$$+ r^{-3}\{V_2'P_2 + V_{2,1}'P_{2,1}\cos(\phi - \beta_1') + V_{2,2}'P_{2,2}\cos(2\phi - \beta_2')\},$$

and by surface forces such as (5) over both surfaces; it being assumed that ε', ε_1', ε_2' are small as well as ε, ε_1, ε_2. Special cases of this shell problem are easily enough worked out. It is expedient, however, when the shell is thin, to look carefully to the convergence of the results obtained, unless the two surfaces of the shell are so nearly similar and similarly situated that the maximum differences between the thickness at different points are small compared to the mean thickness.

§59. Before proceeding to illustrate the application of the previous solution, it is desirable to consider the shape of the surface (3) a little more closely and to explain how bodily forces of the kind we have considered may originate.

Since squares of ε, etc., are neglected, (3) may be written

$$a^2 = r^2 \left[1 - \varepsilon (3 \cos^2 \theta - 1) - 2\varepsilon_1 \sin\theta \cos\theta \cos(\phi - \alpha_1) - 2\varepsilon_2 \sin^2\theta \cos(2\phi - \alpha_2)\right],$$

or

$$\begin{aligned} a^2 = x^2 (1 + \varepsilon - 2\varepsilon_2 \cos\alpha_2) + y^2 (1 + \varepsilon + 2\varepsilon_2 \cos\alpha_2) + z^2 (1 - 2\varepsilon) \\ - 2yz\varepsilon_1 \sin\alpha_1 - 2zx\varepsilon_1 \cos\alpha_1 - 4xy\varepsilon_2 \sin\alpha_2. \quad (293) \end{aligned}$$

The surface is thus the most general form of nearly spherical ellipsoid. Comparing (293) with the Cartesian equation of any such ellipsoid we obtain the values of ε, ε_1, ε_2, α_1 and α_2.

If, for instance, the ellipsoid be referred to its principal axes,

$$\varepsilon_1 = 0 = \alpha_2, \qquad (294)$$

and, supposing the Cartesian equation to be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

$$\left.\begin{aligned} a'/a &= 1 - \frac{1}{2}\varepsilon + \varepsilon_2, \\ b'/a &= 1 - \frac{1}{2}\varepsilon - \varepsilon_2, \\ c'/a &= 1 + \varepsilon. \end{aligned}\right\} \qquad (295)$$

The reason why the ellipsoid was not referred to its principal axes in our general problem was that these may in many cases be far from convenient axes to refer the applied forces to.

§60. As showing how the bodily forces might originate, let us take a nearly spherical ellipsoid acted on by a very distant gravitating body.

Let M' be the mass of this body, R its distance, and let its angular coordinates referred to axes at the centre of the ellipsoid be θ', ϕ'. Then at a point

r, θ, ϕ in the ellipsoid, the gravitation potential due to the distant body is to a second approximation

$$\frac{M'}{R}+\frac{M'r^2}{R^3}\Big[\frac{3\cos^2\theta'-1}{2}\cdot\frac{3\cos^2\theta-1}{2}+\frac{3}{2}\sin 2\theta'\cos\phi'\sin\theta\cos\theta\cos\phi$$
$$+\frac{3}{2}\sin 2\theta'\sin\phi'\sin\theta\cos\theta\sin\phi+\frac{3}{4}\sin^2\theta'\cos 2\phi'\sin^2\theta\cos 2\phi$$
$$+\frac{3}{4}\sin^2\theta'\sin 2\phi'\sin^2\theta\sin 2\phi\Big].^{*}$$

The variable part becomes identical with the expression (4) for the potential of the bodily forces when we put

$$\left.\begin{aligned}\frac{3\cos^2\theta'-1}{2}\,\frac{M'}{R^3}&=V_2,\\ \frac{3}{2}\sin 2\theta'\,\frac{M'}{R^3}&=V_{2,\,1},\\ \frac{3}{4}\sin^2\theta'\,\frac{M'}{R^3}&=V_{2,\,2},\\ \phi'=\beta_1&=\frac{1}{2}\,\beta_2.\end{aligned}\right\}\qquad(296)$$

Our solution thus gives the "tides" on the equilibrium theory in the nearly spherical ellipsoid due to the distant body, on the hypothesis that the ellipsoid may be regarded as an isotropic elastic solid.

§61. If the attracted body be spherical, the forces arising from the second harmonic terms are in equilibrium; otherwise there is precession and nutation.

The couples about the principal axes of the ellipsoid being given in works on astronomy, may conveniently be used to test the accuracy of our results (285) to (287).

Thus if A, B, C be the principal moments of inertia of the ellipsoid, and α, β, γ the direction angles of the distant body referred to the principal axes, the moment about the axis of y in the direction z to x is†

$$-\frac{3M'}{R^3}(C-A)\cos\alpha\cos\gamma\,.$$

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[*] See Todhunter's "The Functions of Laplace . . . . ," Art. 169.

[†] Routh's "Rigid Dynamics," p. 497, 3d edition.

Taking M for the mass, a', b', c' for the principal semi-axes of the ellipsoid, the component couples thus become

about the axis of x, $-\frac{3}{5}\frac{MM'}{R^3}(c'^2-b'^2)\cos\beta\cos\gamma$,

" " y, $-\frac{3}{5}\frac{MM'}{R^3}(a'^2-c'^2)\cos\gamma\cos\alpha$,

" " z, $-\frac{3}{5}\frac{MM'}{R^3}(b'^2-a'^2)\cos\alpha\cos\beta$.

Now by (295) we have to a first approximation

$$\left.\begin{aligned}(a'^2-b'^2)/a^2&=4\epsilon_2\quad,\\(a'^2-c'^2)/a^2&=2\epsilon_2-3\epsilon,\\(c'^2-b'^2)/a^2&=2\epsilon_2+3\epsilon,\end{aligned}\right\}\qquad(297)$$

where a is the mean radius; also

$$\cos\gamma=\cos\theta',\quad\cos\alpha=\sin\theta'\cos\phi',\quad\cos\beta=\sin\theta'\sin\phi'.$$

Thus the component couples are

about the axis of x, $-\frac{3}{5}\frac{MM'a^2}{R^3}(3\epsilon+2\epsilon_2)\cos\theta'\sin\theta'\sin\phi'$,

" " y, $-\frac{3}{5}\frac{MM'a^2}{R^3}(2\epsilon_2-3\epsilon)\cos\theta'\sin\theta'\cos\phi'$,

" " z, $\frac{6}{5}\frac{MM'a^2}{R^3}\epsilon_2\sin^2\theta'\sin 2\phi'$.

Finally, substituting $4\pi\rho a^3/3$ for M and using (294) and (296), we find these components become

$$-\frac{4}{15}\pi\rho a^5(3\epsilon+2\epsilon_2)\sin\beta_1 V_{2,1},$$

$$\frac{4}{15}\pi\rho a^5(3\epsilon-2\epsilon_2)\cos\beta_1 V_{2,1},$$

$$\frac{32}{15}\pi\rho a^5\epsilon_2\sin\beta_2 V_{2,2}.$$

These expressions agree, as they ought, with (285), (286) and (287), since ϵ_1 and α_2 vanish and none of the Θ constants exist.

§62. As the examples by which we are about to illustrate the general solution are particular cases of the gravitational problem just referred to, one point in connection with it claims our attention, viz. that as the nearly spherical ellipsoid is not in equilibrium under the action of the distant body, the application of our solution to it may be questioned. The ellipsoid may, however, be in equilibrium under the action of a series of distant bodies, and since the "uncompensated" terms depend only on the statical resultant, they would in such a case entirely disappear and our solution would give correctly the effects to be attributed to the particular distant body under consideration.

Even if the ellipsoid be not in equilibrium, it will probably in general be convenient to classify the strain under two heads, one part being the strain given by the elastic theory on the hypothesis that equilibrium is maintained, and the other, if such exists, answering to the actual state of motion.

When rotation exists about an axis, whether uniform or not, strains corresponding to the "centrifugal" forces obviously exist. Uniform motion in a straight line is perhaps the only species of motion in which strains may not be expected to arise. An example will show the difficulty of the question in actual practice. Suppose what would, in the absence of external action, be a homogeneous spherical ball to start falling from rest towards the earth. Since "gravity" increases as the distance from the earth diminishes, the portion of the ball nearest the earth is always acted on by a greater gravitational force than the portion more remote, and would always be moving faster if there were no cohesional force. Thus in reality the ball must be in a state of tension in lines parallel to the vertical diameter, and the amount of this tension will vary as the ball approaches the earth. To determine exactly what the strain is at every point at every instant seems not unlikely to be a difficult business. It may be approximately the same as that given by a theory which treats the ball as at rest and acted on by a force given by the second harmonic term in the instantaneous value of the gravitational potential. There may, however, be oscillations set up in the ball, depending partly on the exact nature of the initial conditions, and an equilibrium theory can take no account of these.

§63. The first example we shall consider is a nearly spherical spheroid under the gravitational influence of a distant body in its polar axis. Let M' denote the mass, R the distance of the distant body, and let the equation to the spheroid be

$$r = a(1 + \varepsilon P_2). \tag{298}$$

In our general formulæ we have

$$V_2 = M'/R^3,$$
$$V_{2,1} = V_{2,2} = 0.$$

Substituting these values we get,

$$\left.\begin{aligned}
&\text{from (245) and (246),}\\
&Y_0 = \frac{24\,(4m-n)}{5\,(3m-n)(19m-5n)}\,\varepsilon a^2\rho\,\frac{M'}{R^3}\,;\\
&\text{from (231), (247), (248) and (249),}\\
&Y_2 + \delta Y_2 = \frac{3}{19m-5n}\left\{\frac{7m-n}{m+n} - 2\varepsilon\,\frac{5m-3n}{19m-5n}\right\}\frac{\rho M'}{R^3}\,,\\
&Z_2 + \delta Z_2 = \frac{1}{n\,(19m-5n)}\left\{2\,(4m-n) + \varepsilon\,\frac{89m^2-48mn+7n^2}{19m-5n}\right\}\frac{a^2\rho M'}{R^3}\,;\\
&\text{from (261), (262) and (263),}\\
&Y_4 = 0,\\
&Z_4 = \frac{24\,(4m-n)}{35n\,(19m-5n)}\,\varepsilon\rho\,\frac{M'}{R^3}\,;
\end{aligned}\right\}\qquad(299)$$

all the other coefficients vanish.

These results may also be derived from equations (74) to (79) of Section I.

Substituting for the constants in the general solution (1), (2), (3) of Section I, we find

$$\Delta = \frac{2\rho M'R^{-3}}{19m-5n}\left[\frac{12}{5}\,\varepsilon a^2\,\frac{4m-n}{3m-n} + \left\{1 - 3\varepsilon\,\frac{5m-3n}{19m-5n}\right\}r^2P_2\right], \qquad (300)$$

$$\begin{aligned}
u = \frac{\rho M'R^{-3}}{n\,(19m-5n)}\Bigg[&\frac{8}{5}\,\varepsilon a^2r\,\frac{n\,(4m-n)}{3m-n} + P_2\Bigg\{2\,(4m-n)\,a^2r - (3m-n)\,r^3\\
&+ \frac{\varepsilon}{19m-5n}\left((89m^2-48mn+7n^2)\,a^2r + \frac{6}{7}\,(m-n)(5m-3n)\,r^3\right)\Bigg\}\\
&+ \frac{24}{35}\,\varepsilon r^3P_4\,(4m-n)\Bigg],
\end{aligned} \qquad (301)$$

$$\begin{aligned}
v = \frac{\rho M'R^{-3}}{n\,(19m-5n)}\,\frac{d}{d\theta}\Bigg[&r^3P_2\left\{-\frac{1}{2}\,(5m-n) + \frac{1}{7}\,\varepsilon\,\frac{(5m+2n)(5m-3n)}{19m-5n}\right\}\\
&+ a^2rP_2\left\{4m-n+\frac{1}{2}\,\varepsilon\,\frac{89m^2-48mn+7n^2}{19m-5n}\right\} + \frac{6}{35}\,\varepsilon r^3P_4\,(4m-n)\Bigg].
\end{aligned} \qquad (302)$$

§64. The change in shape is found from the surface values of u and v, obtained by substituting $a\,(1 + \varepsilon P_2)$ for r.

Calling these values u_s and v_s, we have

$$u_s = \frac{a^3 \rho M' R^{-3}}{n\,(19m - 5n)} \Big[-\frac{3}{5}\,\varepsilon\,\frac{m^2 - 12mn + 3n^2}{3m - n}$$

$$+ P_2 \left\{ 5m - n + \frac{3}{7}\,\varepsilon\,\frac{205m^2 - 112mn + 19n^2}{19m - 5n} \right\} + \frac{6}{35}\,\varepsilon P_4\,(13m - n) \Big], \qquad (303)$$

$$v_s = \frac{a^3 \rho M' R^{-3}}{n\,(19m - 5n)}\, \frac{d}{d\theta} \Big[P_2 \left\{ \frac{1}{2}\,(3m - n) + \frac{2}{7}\,\varepsilon\,\frac{135m^2 - 73mn + 8n^2}{19m - 5n} \right\}$$

$$- \frac{3}{70}\,\varepsilon P_4\,(5m + n) \Big]. \qquad (304)$$

If $(u)_p$ and $(u)_e$ be the increments in the polar and equatorial semi-diameters respectively,

$$(u)_p = \frac{a^3 \rho M' R^{-3}}{n\,(19m - 5n)} \left[5m - n + \frac{24}{5}\,\varepsilon\,\frac{m\,(79m^2 - 37mn + 4n^2)}{(3m - n)(19m - 5n)} \right], \qquad (305)$$

$$(u)_e = \frac{a^3 \rho M' R^{-3}}{n\,(19m - 5n)} \Big[-\frac{1}{2}\,(5m - n)$$

$$- \frac{3}{20}\,\varepsilon\,\frac{637m^3 - 1491m^2 n + 667mn^2 - 85n^3}{(3m - n)(19m - 5n)} \Big]. \qquad (306)$$

When terms in ε are neglected the increase in the polar diameter is exactly double the diminution in the equatorial, and to this degree of approximation the volume is unaltered. These first approximation results are, of course, obtainable from the solution for the perfect sphere.

When terms in ε are retained we find, assuming $m \not< n$, that the polar diameter is more or less increased than in a sphere of the same volume according as the spheroid is prolate or oblate.

§65. When ε is of the same order of magnitude as it would be if (298) represented the strained form of what but for the strain would be a spherical surface, the alteration of form is easily found. We have only to put

$$r = a\,(1 + \varepsilon P_2) \pm u_s,$$

taking the $+$ or the $-$ sign according as we regard (298) as the unstrained and seek the strained surface, or regard it as the strained and seek the unstrained surface.

If, however, ε, though small, be large compared to a surface strain, we must take account of the alteration in the inclination of the radius vector to the axis. To the present degree of approximation we may take the increment of this inclination as $a^{-1}v_s'$, where v_s' represents the principal term in the surface value of v, or

$$v_s' = \frac{a^3\rho M'R^{-3}(3m-n)}{2n(19m-5n)}\frac{dP_2}{d\theta}.$$

If (298) represents the unstrained surface, then the surface after strain is given by

$$r = a\left\{1+\varepsilon\frac{3\cos^2(\theta-a^{-1}v_s')-1}{2}\right\}+u_s.$$

The increment in a radius vector amounts to

$$u_s-\varepsilon v_s'\frac{dP_2}{d\theta} = u_s - \frac{\varepsilon a^3(\rho M'R^{-3})(3m-n)}{2n(19m-5n)}\left(\frac{dP_2}{d\theta}\right)^2.$$

§66. Supposing, for instance, we wished to calculate the increment $\delta\bar{v}$ in the volume from the alteration of the surface, we should, as terms in ε^2 are neglected, put

$$\delta\bar{v} = \int_0^\pi\int_0^{2\pi}\left(u_s-\varepsilon v_s\frac{dP_2}{d\theta}\right)r^2\sin\theta\,d\theta\,d\phi.$$

Substituting the values of u_s, v_s and r, we get

$$\delta\bar{v} = \frac{a^5\rho M'R^{-3}}{n(19m-5n)}\int_0^\pi\int_0^{2\pi}\sin\theta\,d\theta\,d\phi\,(1+2\varepsilon P_2)\Big[-\frac{3}{5}\varepsilon\frac{m^2-12mn+3n^2}{3m-n}$$
$$+(5m-n)P_2-\frac{1}{2}\varepsilon(3m-n)\left(\frac{dP_2}{d\theta}\right)^2+\text{terms in } \varepsilon P_2 \text{ and } \varepsilon P_4\Big].$$

The terms in εP_2 and εP_4 inside the square bracket clearly lead to no terms of the order retained; and referring to (7) and (8), and to the well-known result of integrating spherical harmonics, we easily find

$$\delta\bar{v} = \frac{a^5\rho M'R^{-3}}{n(19m-5n)}4\pi\varepsilon\left[-\frac{3}{5}\frac{m^2-12mn+3n^2}{3m-n}+\frac{2}{5}(5m-n)-\frac{3}{5}(3m-n)\right],$$

or

$$\delta\bar{v} = \frac{8}{5}\frac{\pi\varepsilon a^5\rho M'R^{-3}}{3m-n}. \tag{307}$$

The volume is thus increased or diminished according as the spheroid is prolate or oblate.

We may also obtain (307) by integrating the value (300) of Δ throughout the volume bounded by (298).

A verification quite independent of the present solution is obtained from the general formula given in §25. In the present instance this assumes the form

$$(3m - n)\,\delta\bar{v} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a(1+\epsilon P_2)} r\,\frac{dV}{dr}\,.\,r^2 \sin\theta\, d\phi\, d\theta\, dr,$$

where $V = M' r^2 R^{-3} P_2$.

The result (307) follows in a line or two.

This is a satisfactory confirmation, so far as it goes, of the accuracy of our solution. The only terms in ϵ it actually tests are, however, those independent of θ in the values of Δ and u.

§67. As an example where there is not symmetry about an axis, take the case of the nearly spherical ellipsoid

$$r = a\,(1 + \epsilon P_2 + \epsilon_2 P_{2,\,2} \cos 2\phi) \tag{308}$$

under the gravitating action of a body of mass M' at a great distance R, in the direction $\phi = 0$, $\theta = \pi/2$.

The harmonic terms in the potential are

$$(r^2 R^{-3} M')\left\{-\frac{1}{2}P_2 + \frac{3}{4}P_{2,\,2}\cos 2\phi\right\};$$

thus in our general solution we are to put

$$\left.\begin{aligned} V_2\Big/\left(-\tfrac{1}{2}\right) = V_{2,\,2}\Big/\left(\tfrac{3}{4}\right) = M'/R^3,\\ \alpha_2 = \beta_2 = 0. \end{aligned}\right\} \tag{309}$$

The arbitrary constants which do not vanish have the following values:

$$\left.\begin{aligned}
Y_0 &= \frac{12(4m-n)}{5(3m-n)(19m-5n)}(2\varepsilon_2-\varepsilon)\rho a^2R^{-3}M',\\
Y_2+\delta Y_2 &= -\frac{3}{2}\,\frac{\rho R^{-3}M'}{19m-5n}\left[\frac{7m-n}{m+n}-2(\varepsilon+2\varepsilon_2)\frac{5m-3n}{19m-5n}\right],\\
Z_2+\delta Z_2 &= \frac{-\rho a^2R^{-3}M'}{n(19m-5n)}\left[4m-n+\frac{1}{2}(\varepsilon+2\varepsilon_2)\frac{89m^2-48mn+7n^2}{19m-5n}\right],\\
Y_{2,2}+\delta Y_{2,2} &= \frac{3}{2}\,\frac{\rho R^{-3}M'}{19m-5n}\left[\frac{3}{2}\,\frac{7m-n}{m+n}+(3\varepsilon-2\varepsilon_2)\frac{5m-3n}{19m-5n}\right],\\
Z_{2,2}+\delta Z_{2,2} &= \frac{1}{2}\,\frac{\rho a^2R^{-3}M'}{n(19m-5n)}\left[3(4m-n)\right.\\
&\qquad\left.-\frac{1}{2}(3\varepsilon-2\varepsilon_2)\frac{89m^2-48mn+7n^2}{19m-5n}\right],\\
Z_4 &= \frac{4(4m-n)}{35n(19m-5n)}(\varepsilon_2-3\varepsilon)\rho R^{-3}M',\\
Z_{4,2} &= \frac{4m-n}{2n(19m-5n)}(3\varepsilon-2\varepsilon_2)\rho R^{-3}M',\\
Z_{4,4} &= \frac{4m-n}{2n(19m-5n)}\varepsilon_2\rho R^{-3}M',\\
W'_{3,2} &= \frac{m}{4n(19m-5n)}(3\varepsilon+2\varepsilon_2)\rho R^{-3}M'.
\end{aligned}\right\}\quad(310)$$

Substituting the above values of the arbitrary constants along with the values of V_2 and $V_{2,2}$ from (309) in (1), (2), (3) and (4) of Section I, we get the dilatation and displacements.

Since there is no Y constant with suffix 4, there is no surface harmonic of the 4th degree in the expression for the dilatation, but unless both ε and ε_2 vanish there will exist at least two Z constants with suffix 4, and so surface harmonics of the fourth degree will present themselves in the values of the displacements.

Since Y_0 exists unless

$$2\varepsilon_2=\varepsilon, \qquad (311)$$

we see by reference to (297) that there will be a term in the dilatation independent of the angular coordinates, and a corresponding pure radial displacement, unless

$$b'^2+c'^2=2a'^2; \qquad (312)$$

or the square on the principal diameter on which the distant body lies must be the mean of the squares on the three principal diameters if the pure radial displacement is to vanish.

Again, by reference to the value of $W'_{3,2}$ we see that a pure transverse displacement will exist unless

$$3\varepsilon + 2\varepsilon_2 = 0, \tag{313}$$

a relation signifying that the surface is a spheroid of revolution about the axis on which the distant body lies.

§68. The expressions for the displacements answering to (310) are rather long, so we shall confine our attention to the radial displacement as the most interesting. For it we have

$$\begin{aligned} u = \frac{\rho M' R^{-3}}{19m - 5n} \Big[& \frac{4}{5} \frac{4m - n}{3m - n} (2\varepsilon_2 - \varepsilon) a^2 r \\ & + \frac{1}{2n} \Big\{ (4m - n) a^2 r - \frac{1}{2} (3m - n) r^3 \Big\} (-2P_2 + 3P_{2,2} \cos 2\phi) \\ & - \frac{1}{2n(19m - 5n)} \Big\{ \frac{1}{2} (89m^2 - 48mn + 7n^2) a^2 r \\ & + \frac{3}{7} (m - n)(5m - 3n) r^3 \Big\} \{ 2(\varepsilon + 2\varepsilon_2) P_2 + (3\varepsilon - 2\varepsilon_2) P_{2,2} \cos 2\phi \} \\ & + \frac{4m - n}{n} r^3 \Big\{ \frac{4}{35} (\varepsilon_2 - 3\varepsilon) P_4 + \frac{1}{2} (3\varepsilon - 2\varepsilon_2) P_{4,2} \cos 2\phi \\ & + \frac{1}{2} \varepsilon_2 P_{4,4} \cos 4\phi \Big\} \Big]. \end{aligned} \tag{314}$$

Putting $\varepsilon_2 = 0$, we have the case of a spheroid with a distant attracting body in its equatorial plane. In this case the radial displacement at the surface is given by

$$\begin{aligned} u_s = \frac{\rho a^3 M' R^{-3}}{n(19m - 5n)} \Big[& \frac{1}{4} (5m - n)(-2P_2 + 3P_{2,2} \cos 2\phi) \\ & + 3\varepsilon \Big\{ \frac{1}{10} \frac{m^2 - 12mn + 3n^2}{3m - n} - \frac{205m^2 - 112mn + 19n^2}{28(19m - 5n)} (2P_2 + 3P_{2,2} \cos 2\phi) \\ & - (13m - n) \Big(\frac{1}{35} P_4 - \frac{1}{8} P_{4,2} \cos 2\phi \Big) \Big\} \Big]. \end{aligned} \tag{315}$$

It will be noticed that the value of $-2P_2 + 3P_{2,2} \cos 2\phi$ is the same for $\theta = 0$ as for $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$; also that the values of both $2P_2 + 3P_{2,2} \cos 2\phi$ and $\frac{1}{35} P_4 - \frac{1}{8} P_{4,2} \cos 2\phi$ are the same for $\theta = 0$ as for $\theta = \pi/2$, $\phi = 0$. Thus the principal terms in u_s have the same value at the ends of the polar diameter and the equatorial diameter perpendicular to that on which the distant body lies, whereas the subsidiary terms have the same value at the ends of the polar diameter and the equatorial diameter on which the body lies.

Employing $(u)_p$, $(u)_e$ and $(u)_{e'}$ for the increments of the polar semi-diameter, the equatorial semi-diameter on which the distant body lies, and the perpendicular semi-diameter respectively of the spheroid, we find

$$(u)_{\mathrm{p}} = \frac{-\rho a^3 M' R^{-3}}{n(19m-5n)}\left\{\frac{1}{2}(5m-n) + \frac{12}{5}\varepsilon\frac{m(79m^2-37mn+4n^2)}{(3m-n)(19m-5n)}\right\}, \tag{316}$$

$$(u)_e = \frac{\rho a^3 M' R^{-3}}{n(19m-5n)}\left\{5m-n-\frac{12}{5}\varepsilon\frac{m(79m^2-37mn+4n^2)}{(3m-n)(19m-5n)}\right\}, \tag{317}$$

$$(u)_{e'} = \frac{-\rho a^3 M' R^{-3}}{n(19m-5n)}\left\{\frac{1}{2}(5m-n) - \frac{3}{20}\varepsilon\frac{1901m^3-2083m^2n+731mn^2-85n^3}{(3m-n)(19m-5n)}\right\}, \tag{318}$$

Comparing (316) with (305) we see that the increase in the polar diameter of a nearly spherical spheroid due to the attraction of a distant body situated on that diameter produced, is, as far as terms in ε, exactly double the diminution experienced by that diameter when the attracting body is at the same distance away in the equatorial plane.

Again, from (317) and (318) we find

$$(u)_e + (u)_{e'} = \frac{\rho a^3 M' R^{-3}}{n(19m-5n)}\left\{\frac{1}{2}(5m-n) + \frac{3}{20}\varepsilon\frac{637m^3-1491m^2n+667mn^2-85n^3}{(3m-n)(19m-5n)}\right\},$$

which, with a change of sign, is precisely the expression occurring on the right-hand side of (306).

We accordingly have the curious result that when a distant body is in the equator of a spheroid, the algebraical sum of the increments to the equatorial diameter on which the distant body lies and the perpendicular equatorial diameter is numerically equal, so far as terms in ε, to the reduction in an equatorial diameter when the body is the same distance away on the polar axis.

Assuming $m-n$ positive, i. e. Poisson's ratio positive, we find from (316) and (318) that when a nearly spherical spheroid is attracted by a distant body in its equatorial plane, the reduction of the equatorial diameter perpendicular to that on which the body lies is greater or less than the reduction of the polar diameter according as ϵ is negative or positive, i. e. according as the spheroid is oblate or prolate.

§69. Taking the general case of the surface (308), we find, by integrating the value of Δ throughout the volume, that the increment $\delta\bar{v}$ in volume is given by

$$\delta\bar{v} = \frac{4}{5}\,\frac{\pi(2\epsilon_2 - \epsilon)}{3m - n}\,\rho a^5 M' R^{-3}. \tag{319}$$

This result is easily verified by the second method of §25. It shows that the change in volume is zero only when (311) holds, and the general law is that the volume is increased or diminished according as the square of the diameter along which the distant body lies is greater or less than the mean of the squares on the two other principal diameters.

§70. The general case of a nearly spherical spheroid acted on by a distant body, the diameter through which makes an angle θ' with the axis of symmetry, comes of course within the range of the present method. It presents various points of interest, but I shall only refer to the expression for the increment of volume, viz.

$$\delta\bar{v} = \frac{4}{5}\,\frac{\pi\epsilon(3\cos^2\theta' - 1)}{3m - n}\,\rho a^5 M' R^{-3}. \tag{320}$$

It shows that in an oblate spheroid, for which ϵ is negative, the volume is increased or diminished according as θ' is greater or less than θ_0, where

$$\theta_0 = \cos^{-1}(\sqrt{1/3}).$$

For a prolate spheroid the exact reverse holds.

§71. If for a moment we assumed our formulæ to apply to the action of the sun on the earth,* we should have the earth's volume increased above what it would be but for the sun's attraction, and this increase would be the greater the nearer the sun to the equator. From this cause the volume would be a maximum at the two equinoxes and a minimum at the two solstices, going through a complete cycle in six months.

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* The earth in its surface layers is far from isotropic, and what its constitution may be except close to the surface is unknown, so that the subsequent remarks are purely speculative.

§72. Such an hypothesis may possess an interest for physicists in quest of agencies to account for periodic fluctuations in terrestrial phenomena. Its possible bearing on such phenomena as earthquakes, naturally associated with ruptures produced by internal stresses, is too obvious to require pointing out; but as the possible connection with magnetic phenomena is a little less obvious, it may be briefly indicated.

Let us consider what would happen if the nearly spherical spheroid were of magnetized iron,* the distant body for simplicity being supposed non-magnetic. The magnetization of iron varies with the state of strain, whether the iron is being magnetized by external influences or is possessed only of "residual" magnetism, the variation being in some cases very considerable. Thus as the state of strain alters, so also do the external manifestations of the spheroid's magnetism. Now suppose the spheroid to have a diurnal rotation round its axis, and for simplicity regard the strain at any instant as given by the equilibrium theory. Then it is obvious from the occurrence of cos 2ϕ in (315) that the intensity of magnetic phenomena at any point on the surface will show a semi-diurnal variation, since the strain at every point throughout the spheroid has a semi-diurnal variation.†

If the distant body were at a constant distance away in the equatorial plane there would be only a cycle with a semi-diurnal period.‡ But if the diameter pointing to the distant body alter its inclination to the equator, going through a complete cycle in one year, there will be, cf. (320), a corresponding fluctuation of strain at every point with a semi-annual period, and so a variation of magnetic phenomena with a semi-annual period.

**Iron* is specified merely for brevity. The permeability of other magnetic substances is also affected by strain. It is not intended to imply that the earth's interior is of iron.

† [Sept. 10.—Since the above was written, a letter from Prof. Cleveland Abbe, who was unaware I had been considering the subject, has recalled to my recollection the fact that this suggestion has been already made by Maxwell, "Electricity and Magnetism," Vol. II, Art. 474.]

‡ [Sept. 10.—The occurrence of cos 4ϕ in (314) shows that in a nearly spherical *ellipsoid* there would also be a cycle, of small intensity, with a six hour period.]

If any action of this kind may be assigned to the sun, it is of course equally to be expected from the moon. If a very decided influence is exerted by the sun and no appreciable influence by the moon, the explanation cannot account for more than an insignificant portion of the phenomena and some other cause must be sought for. This need not be any direct magnetic influence from the sun, as the heating* effect might be the exciting cause. It affects the superficial strains in addition to altering the electric conductivity of the air and the earth's surface layers.

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*[Sept. 10.—The effects of *light* may also be important. See a paper by Elster and Geitel, Wien Sitz. 101, Abth. II, p. 703, 1892.]